Separation Axiom (T₀) on Fuzzy Bitopological Space in Quasi-coincidence Sense

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ABSTRACT

In this paper, we introduce two notions of T₀ separation on fuzzy bitopological space using quasi-coincidence sense and establish relations among ours and other counterparts. We show that our notions satisfy “good extension”, “hereditary”, “productive”, “projective” properties and observe that our concepts are preserved under one-one, onto, fuzzy open and fuzzy continuous mappings. Finally, we discuss initial and final fuzzy bitopological spaces on the mentioned concepts.

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1. Introduction

The first concept of fuzzy sets introduced by Zadeh [1] in 1965. By using this concept Chang [2] defined fuzzy topological spaces in 1968. Separation axioms [3, 4, 5] are important parts in fuzzy topological spaces. Also many researchers have contributed various types of separation axioms [6, 7, 8] on fuzzy bitopological space which is introduced by Kandil and El-Shafee [9] in 1991. Among those axioms, T₀ type separation on fuzzy bitopological space is one and it has been introduced earlier by Abu Sufia et al. [10], Nouh [11], Amin et al. [12] and others.

The purpose of this paper is to further contribute to the development of fuzzy bitopological spaces especially on fuzzy T₀ bitopological spaces. In the present paper, fuzzy T₀ bitopological space is defined by using quasi-coincidence sense and it is showed that the good extension property is satisfied on our notions. In the next section of this paper, it is also showed that the hereditary, order preserving, productive, and projective properties hold on the new concepts. Initial and final fuzzy bitopologies are discussed also.

2. Fuzzy T₀ Bitopological Space

In this section, we discuss about our notions and findings. Some well-known properties are discussed here by using our concepts. Here X and Y always denote nonempty sets.

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Definition 1: A fuzzy bitopological space \((X, s, t)\) is called
(a) [10] \(\text{FPT}_0(i)\) if and only if for any pair \(x_m, y_n \in S(X)\) with \(x \neq y\), there exists \(u \in (s \cup t)\) such that \(x_m \in u, y_n \notin u\), or there exists \(v \in (s \cup t)\) such that \(y_n \in v, x_m \notin v\).
(b) \(\text{FPT}_0(ii)\) if and only if for any pair \(x_m, y_n \in S(X)\) with \(x \neq y\), there exists \(u \in (s \cup t)\) such that \(x_m \notin u, y_n \in u = 0\) or there exists \(v \in (s \cup t)\) such that \(y_n \notin v, x_m \in v = 0\).
(c) \(\text{FPT}_0(iii)\) if and only if for any pair \(x_m, y_n \in S(X)\) with \(x \neq y\), there exists \(u \in (s \cup t)\) such that \(x_m \notin u, y_n \notin u\), or there exists \(v \in (s \cup t)\) such that \(y_n \notin v, x_m \notin v\).
(d) [10] \(\text{FPT}_0(iv)\) if and only if for any pair of distinct fuzzy points \(p, q \in X\), there exists a fuzzy set \(u \in (s \cup t)\) such that \(p \notin u, q \notin u = 0\), or \(q \in u, p \notin u = 0\).
(e) [11] \(\text{FPT}_0(v)\) if and only if for any pair \(x_m, y_n \in S(X)\) with \(x \neq y\), there exists \(u \in s \) or \(u \in t\) which is \(Q\)-nbd of one of the fuzzy singletons and not quasi-coincident with other.
(f) [10] \(\text{FPT}_0(vi)\) if and only if for any pair \(x_m, y_n \in S(X)\) with \(x \neq y\), there exists \(u \in (s \cup t)\) such that \(x_m \in u, u \subseteq (y_n)^c\), or \(y_n \in v, u \subseteq (x_m)^c\).
(g) [13] \(\text{FPT}_0(vii)\) if and only if for all \(x, y \in X\) with \(x \neq y\), there exists \(u \in (s \cup t)\) such that \(u(x) = 1, u(y) = 0\) or \(u(y) = 1, u(x) = 0\).

Example 1: Let \(X = \{x, y\}, u \in I^X\) with \(u(x) = 1, u(y) = 0\) and \(t\) be the fuzzy topology on \(X\) generated by \{0, u, 1\} and \(s\) be the fuzzy topology on \(X\) generated by \{constants\}. Also, let \(x_m, y_n \in S(X)\) with \(x \neq y\). Then \(u(x) + m > 1\) and \(u(y) + s \leq 1\). Thus, \(x_m \notin u\) and \(y_n \notin u\) is quasi-coincidence with \(u\) and \(y_n\) is not quasi-coincidence with \(u\). Hence \((X, s, t)\) is \(\text{FPT}_0(iii)\).

Theorem 1: Let \((X, s, t)\) be a fuzzy bitopological space and \((X, s \cup t)\) be a fuzzy topological space. Then \((X, s, t)\) is \(\text{FPT}_0\) if and only if \((X, s \cup t)\) is \(\text{FPT}_0\).

Proof: Let \((X, s, t)\) be \(\text{FPT}_0\). Since \(s \subseteq (s \cup t)\) and \(t \subseteq (s \cup t)\), it follows that \((X, s \cup t)\) is \(\text{FPT}_0\).

Conversely, let \((X, s \cup t)\) be \(\text{FPT}_0\) and \(x_m, y_n\) are fuzzy singletons in \(X\) with \(x \neq y\). Then \(\exists u \in (s \cup t)\) such that \(x_m \notin u\) and \(y_n \notin u\) is \(\text{FPT}_0(iii)\). Hence \((X, s, t)\) is \(\text{FPT}_0\).

Theorem 2: If the fuzzy topological space \((X, s)\) and \((X, t)\) are both \(\text{FPT}_0(f)\), then their corresponding fuzzy bitopological space \((X, s, t)\) is \(\text{FPT}_0(f)\), for \(f = i, iii\). But the converse is not true in general.

Theorem 3: For a fuzzy bitopological space \((X, s, t)\) the following implications are true:

![Figure 2.1: Implications of notions](image)

where \((a), (b), (c), (d), (e), (f)\) and \((g)\) are used, for convenience, instead of \(\text{FPT}_0(i), \text{FPT}_0(ii), \text{FPT}_0(iii), \text{FPT}_0(iv), \text{FPT}_0(v), \text{FPT}_0(vi)\) and \(\text{FPT}_0(vii)\) respectively.

Proof: \((a) \Rightarrow (b)\): Let \((X, s, t)\) be \(\text{FPT}_0(i)\) and \(x_m, y_n\) be fuzzy singletons in \(X\) with \(x \neq y\). Also let \(r > 1 - m\) for \(0 < m < 1\). Since \((X, s, t)\) is \(\text{FPT}_0(i)\), there exists \(u \in (s \cup t)\) such that \(x_r \in u, y_n \notin u\).

Now, \(x_r, u \in u(x) \geq r > 1 - m \Rightarrow u(x) + m > 1 \Rightarrow x_m \notin u\) and \(y_n \notin u \Rightarrow u(y) + 1 \leq 1 \Rightarrow u(y) \leq 1 - 1 = 0 \Rightarrow u(y) = 0 \Rightarrow y_n \notin u = 0\).

It follows that for any fuzzy singletons \(x_m, y_n\) in \(X\) with \(x \neq y\) there exists \(u \in (s \cup t)\) such that \(x_m \notin u, y_n \notin u = 0\). Thus \((X, s, t)\) is \(\text{FPT}_0(ii)\).
(b) ⇒ (d): Let $x_m, y_n$ be distinct fuzzy singletons in $X$ and $0<r<1$ with $r \leq 1-m$. Since $(X, s, t)$ is FPT$_0(ii)$, there exists $u \in s$ such that $x_qu$ and $y_nu = 0$ or there exists $v \in t$ such that $y_nu v$ and $x_rv = 0$.

Now, $x_qu \Rightarrow u(x) + r > 1 \Rightarrow u(x) + 1 - r \geq m \Rightarrow u(x) \geq m \Rightarrow x_m \in u$

It follows that for any fuzzy singletons $x_m, y_n$ in $X$ with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_mu$, $y_nu \Rightarrow 1$. Thus $(X, s, t)$ is FPT$^0(t)$.

(d) ⇒ (e): Let $x_m, y_n$ be distinct fuzzy singletons in $X$ and $0<m_1<1$, $0<n_1<1$ with $m_1 > 1 - m$ and $n_1 > 1 - n$.

Since $(X, s, t)$ is FPT$^0(vi)$ and $x_m, y_n$ are distinct fuzzy points, there exists $u \in (s \cup t)$ such that $u(x) \Rightarrow (s \cup t)$ and $y_nu \Rightarrow u = 0$.

Now, $x_m \in u \Rightarrow u(x) \geq m_1 > 1 - m \Rightarrow u(x) + m > 1 \Rightarrow x_mu$ and $y_nu \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_nq_u$

It follows that for any fuzzy singletons $x_m, y_n$ in $X$ with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_mu$, $y_nu \Rightarrow 1$. Thus $(X, s, t)$ is FPT$^0(u)$.

(e) ⇒ (f): Let $x_m, y_n$ be distinct fuzzy singletons in $X$ and $0<r<1$ with $r \leq 1-m$. Since $(X, s, t)$ is FPT$^0(v)$, there exists $u \in (s \cup t)$ such that $x_qu$ and $y_nq_u$.

Now, $x_qu \Rightarrow u(x) + r > 1 \Rightarrow u(x) + 1 - r > m \Rightarrow u(x) \geq m \Rightarrow x_m \in u$ and $y_nq_u \Rightarrow u(y) + n \leq 1 \Rightarrow n \leq 1 - u(y) = u \subseteq (y_n)^c$.

It follows that for any fuzzy singletons $x_m, y_n$ in $X$ with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_mu, u \subseteq (y_n)^c$.

Therefore, $(X, s, t)$ is FPT$^0(v)$.

(f) ⇒ (a): Let $x_m, y_n$ be distinct fuzzy singletons in $X$. Since $(X, s, t)$ is FPT$^0(vi)$, there exists $u \in (s \cup t)$ such that $x_m \in u$, $u \subseteq (y_n)^c$.

Now, $u \subseteq (y_n)^c \Rightarrow n \leq 1 - u(y) \Rightarrow u(y) + n \leq 1 \Rightarrow y_nq_u$.

It follows that for any fuzzy singletons $x_m, y_n$ in $X$ with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_m \in u$, $u \subseteq (y_n)^c$.

Therefore, $(X, s, t)$ is FPT$^0(i)$.

(b) ⇒ (c): Let $(X, s, t)$ be FPT$^0(ii)$ and $x_m, y_n$ be fuzzy singletons in $X$ with $x \neq y$. Since $(X, s, t)$ is FPT$^0(ii)$, there exists $u \in s$ such that $x_mu$, $y_nu = 0$ or there exists $v \in t$ such that $y_nu v$, $x_mu v = 0$ i.e. we can say that $u \subseteq (s \cup t)$ as $u \subseteq v \subseteq u \cup t$ as $v \subseteq t$. To prove $(X, s, t)$ is FPT$^0(iii)$, it is only needed to prove that $y_nq_u v$ or $x_mq_u v$.

Now, $y_nu \Rightarrow u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_nq_u$.

Thus $(X, s, t)$ is FPT$^0(iii)$.

To show $(c) \Rightarrow (b)$, we shall give a counter example.

Counter example 2: Let $X = \{x, y\}$ and $u, v \in I^X$ be given by $u(x) = 1$, $u(y) = 0.1$, $v(x) = 1$, $v(y) = 0.1$. Let us consider the fuzzy topology $s$ on $X$ generated by $\{0, u, 1\}$ and the fuzzy topology $t$ on $X$ generated by $\{0, v, 1\}$.

For $0 < m \leq 1/2, 0 < n < 0.9$, $u(x) + m > 1 \Rightarrow x_m u$ and $u(y) + n \leq 1 \Rightarrow y_nq_u$.

Thus $(X, s, t)$ is FPT$^0(ii)$ as $u \subseteq (s \cup t)$. But $u(y) \neq 0 \Rightarrow y_nu \neq 0$. Also, $v(x) \neq 0 \Rightarrow x_mu v \neq 0$. Thus $(X, s, t)$ is not FPT$^0(ii)$.

(a) ⇒ (c): As $(d) \Rightarrow (e)$ we can say that $(a) \Rightarrow (c)$.

(g) ⇒ (c): Let $(X, s, t)$ be FPT$^0(vii)$ and $x_m, y_n$ be fuzzy singletons in $X$ with $x \neq y$. Since $(X, s, t)$ is FPT$^0(vii)$, there exists a fuzzy set $u \subseteq (s \cup t)$ such that $u(x) = 1$, $u(y) = 0$ or $u(x) = 0$, $u(y) = 1$. To prove $(X, s, t)$ is FPT$^0(iii)$, it is only needed to prove that $x_mq_u$, $y_nq_u$ or $x_mq_u v$.

Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_mq_u$ and $u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_nq_u$.

Thus $(X, s, t)$ is FPT$^0(iii)$.

To show $(c) \Rightarrow (g)$, we shall give a counter example.

Counter example 3: Let $X = \{x, y\}$ and $u \in I^X$ be given by $u(x) = 1 - y$, $u(y) = 0$, where $y = \frac{m}{2}$ for $m \in (0, 1]$. Let the fuzzy topology $s$ on $X$ generated by $\{0, u, 1\}$ and the fuzzy topology $t$ on $X$ generated by $\{0, 0.5\}$.

Now, $u(x) = 1 - y \Rightarrow u(x) = 1 - \frac{m}{2} \Rightarrow u(x) + \frac{m}{2} = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_mq_u$ and $u(y) = 0 \Rightarrow y_nu \neq 0$. Thus $(X, s, t)$ is FPT$^0(vi)$ as $u \subseteq (s \cup t)$. But $u(x) \neq 1$. Thus $(X, s, t)$ is not FPT$^0(vii)$.

(c) ⇒ (a): As $(b) \Rightarrow (d)$ we can say that $(c) \Rightarrow (a)$.

(g) ⇒ (b): Let $(X, s, t)$ be FPT$^0(vii)$ and $x_m, y_n$ be fuzzy singletons in $X$ with $x \neq y$. Since $(X, s, t)$ is FPT$^0(vii)$, there exists a fuzzy set $u \subseteq (s \cup t)$ such that $u(x) = 1$, $u(y) = 0$ or $u(x) = 0$, $u(y) = 1$. To prove $(X, s, t)$ is FPT$^0(iii)$, it is only needed to prove that $x_mq_u$, $y_nq_u$ or $x_mq_u v$.

Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_mq_u$ and $u(y) = 0 \Rightarrow y_nu \neq 0$. Thus $(X, s, t)$ is FPT$^0(iii)$.

To show $(b) \Rightarrow (g)$, we shall give a counter example.

Counter example 4: Let $X = \{x, y\}$ and $u \in I^X$ be given by $u(x) = 1 - y$, $u(y) = 0$, where $y = \frac{m}{2}$ for $m \in (0, 1]$. Let the fuzzy topology $s$ on $X$ generated by $\{0, u, 1\}$ and the fuzzy topology $t$ on $X$ generated by $\{0, 0.5\}$.

Now, $u(x) = 1 - y \Rightarrow u(x) = 1 - \frac{m}{2} \Rightarrow u(x) + \frac{m}{2} = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_mq_u$ and $u(y) = 0 \Rightarrow y_nu \neq 0$. Thus $(X, s, t)$ is FPT$^0(vi)$ as $u \subseteq (s \cup t)$. But $u(x) \neq 1$. Thus $(X, s, t)$ is not FPT$^0(vii)$.

(b) ⇒ (a): As $(b) \Rightarrow (d)$ and $(d) \Rightarrow (e)$ we can say that $(b) \Rightarrow (a)$. Thus proof is complete.
Theorem 4: Let $(X, S, T)$ be a bitopological space. Then $(X, S, T)$ is PT$_0$ if and only if $(X, \omega(S), \omega(T))$ is FPT$_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

Proof: Let $(X, S, T)$ be a PT$_0$ topological space. We shall prove that $(X, \omega(S), \omega(T))$ is FPT$_0(ii)$. Let $x, y \in X$ with $x \neq y$. Since $(X, S, T)$ is a PT$_0$ topological space hence there exists $U \in (S \cup T)$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Suppose that $x \in U, y \notin U$. From the definition of lower semi continuous function, $1_y \in (\omega(S) \cup \omega(T))$. i.e. $1_y \in \omega(S)$ or $1_y \in \omega(T)$. Then

$$1_y(x) = 1 \Rightarrow 1_y(x) + m > 1 \Rightarrow x_m q 1_y$$

And

$$1_y(y) = 0 \Rightarrow y_n \cap 1_y = 0$$

It follows that there exists $1_y \in \omega(S)$ or $1_y \in \omega(T)$ such that $x_m q 1_y, y_n \cap 1_y = 0$. Hence $(X, \omega(S), \omega(T))$ is FPT$_0(ii)$. Conversely, let $(X, \omega(S), \omega(T))$ be FPT$_0(ii)$. It is required to prove that $(X, S, T)$ be a PT$_0$ topological space. Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(S), \omega(T))$ is FPT$_0(ii)$, we have for any fuzzy singletons $x_m, y_n \in X$, there exists $u \in \omega(S)$ such that $x_m qu, y_n \cap u = 0, or v \in \omega(T)$ such that $y_n q v, x_m \cap v = 0$.

Now, $x_m qu \Rightarrow u(x) + m > 1 \Rightarrow u(x) > 1 - m = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$

And

$$y_n \cap u = 0 \Rightarrow u(y) = 0$$

$$\Rightarrow u(y) + n \leq 1 \Rightarrow u(y) \leq 1 - n = \alpha$$

$$\Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$$

Also, $u^{-1}(\alpha, 1] \in S$. It follows that there exists $u^{-1}(\alpha, 1] \in S$ such that $x \in u^{-1}(\alpha, 1], y \notin u^{-1}(\alpha, 1]$. Hence $(X, S, T)$ be a PT$_0$ topological space.

Now, we shall prove the good extension property of FPT$_0(ii)$. Let $(X, S, T)$ be a PT$_0$ topological space. We shall prove that $(X, \omega(S), \omega(T))$ is FPT$_0(iii)$. Let $x, y \in X$ with $x \neq y$. Since $(X, S, T)$ be a PT$_0$ topological space hence there exists $U \in (S \cup T)$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Suppose that $x \in U, y \notin U$. From the definition of lower semi continuous function, $1_y \in (\omega(S) \cup \omega(T))$. Then

$$1_y(x) = 1 \Rightarrow 1_y(x) + m > 1 \Rightarrow x_m q 1_y$$

And

$$1_y(y) = 0 \Rightarrow 1_y(y) + n \leq 1 \Rightarrow y_n q 1_y$$

It follows that there exists $1_y \in (\omega(S) \cup \omega(T))$ such that $x_m q 1_y, y_n q 1_y$. Hence $(X, \omega(S), \omega(T))$ is FPT$_0(iii)$. Conversely, let $(X, \omega(S), \omega(T))$ be FPT$_0(iii)$. It is required to prove that $(X, S, T)$ be a PT$_0$ topological space. Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(S), \omega(T))$ is FPT$_0(iii)$, for any fuzzy singletons $x_m, y_n \in X$, there exists $u \in (\omega(S) \cup \omega(T))$ such that $x_m qu, y_n qu$.

Now, $x_m qu \Rightarrow u(x) + m > 1 \Rightarrow u(x) > 1 - m = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$

And

$$y_n qu \Rightarrow u(y) + n \leq 1 \Rightarrow u(y) \leq 1 - n = \alpha$$

$$\Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$$

Also, $u^{-1}(\alpha, 1] \in S \cup T$. It follows that there exists $u^{-1}(\alpha, 1] \in (S \cup T)$ such that $x \in u^{-1}(\alpha, 1], y \notin u^{-1}(\alpha, 1]$. Hence $(X, S, T)$ be a PT$_0$ topological space.

Proof for $j = i, iv, v, vi, vii$ is similar to above.

3. Hereditary, Productive and Projective properties

In this section, we describe the hereditary, productive and projective properties (see Appendix A) on our given concepts FPT$_0(j)$ where $j = i, ii, iii$. The first theorem is on hereditary property and the second one is on productive and projective properties.

Theorem 4: Let $(X, s, t)$ be a fuzzy bitopological space and $A \subseteq X$, $s_A = \{u/A: u \in s\}$, $t_A = \{v/A: v \in t\}$ and $(X, s, t)$ is FPT$_0(j)$ then $(A, s_A, t_A)$ is FPT$_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

Proof: We shall prove this theorem for $j = ii$ and remaining is similar.

Let $(X, s, t)$ be FPT$_0(ii)$ and $x_m, y_n$ are fuzzy singletons in $A$ with $x \neq y$. Since $A \subseteq X$, $x_m, y_n$ are also fuzzy singletons in $X$. Also since $(X, s, t)$ is FPT$_0(ii)$, there exists $u \in (s \cup t)$ such that $x_m qu$ and $y_n \cap u = 0$ or $\exists v \in (s \cup t)$ such that $y_n q v$ and $x_m \cap v = 0$. For $A \subseteq X$, we have $u/A \in (s_A \cup t_A)$.

Now, $x_m qu \Rightarrow u(x) + m > 1, x \in X \Rightarrow u(A(x)) + m > 1, x \in A \subseteq X \Rightarrow x_m qu/A$ and $y_n \cap u = 0 \Rightarrow u(y) = 0, y \in X \Rightarrow u/A(y) = 0, y \in A \subseteq X \Rightarrow y_n \cap u/A = 0$

Therefore, $(A, s_A, t_A)$ is FPT$_0(ii)$.

Theorem 5: If $\{(X_i, s_i, t_i): i \in A\}$ is a family of fuzzy bitopological spaces then the product fuzzy bitopological spaces $(\Pi X, \Pi S, \Pi T) = (X, s, t)$ is FPT$_0(j)$ if and only if each coordinate space $(X_i, s_i, t_i)$ is FPT$_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.
Proof: Let for all $i \in A, (X_i, s_i, t_i)$ be $FPT_0(iii)$ space. Let $x_m, y_n$ be fuzzy singletons in $X$ with $x \neq y$. Then $(x_m)_m (y_n)_n$ are fuzzy singletons with $x_i \neq y_i$ for some $i \in A$. Since $(X_i, s_i, t_i)$ is $FPT_0(iii)$, there exists $u_i \in (s_i \cup t_i)$ such that $(x_m)_m u_i (y_n)_n \bar{u}_i$ or $v_i \in (s_i \cup t_i)$ such that $(y_n)_n u_i (x_m)_m \bar{u}_i$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Now, $(x_m)_m u_i \Rightarrow u_i(x) + m > 1 \Rightarrow u_i(\pi_i(x)) + m > 1 \Rightarrow u_i(\pi_i(x)) + m > 1 \Rightarrow x_m q(u_i \circ \pi_i)$. And $(y_n)_n \bar{u}_i \Rightarrow u_i(y) + n > 1 \Rightarrow u_i(\pi_i(y)) + n > 1 \Rightarrow u_i(\pi_i(y)) + n > 1 \Rightarrow y_n \bar{u}_i (u_i \circ \pi_i) = 0$. It follows that there exists $(u_i \circ \pi_i) \in (s_i \cup t_i)$ such that $x_m q(u_i \circ \pi_i), y_n \bar{u}_i (u_i \circ \pi_i) = 0$. Hence, $(X, s, t)$ is $FPT_0(iii)$. Conversely, let the product fuzzy bitopological space $(X, s, t)$ be $FPT_0(iii)$. It is required to prove that for all $i \in A, (X_i, s_i, t_i)$ is $FPT_0(iii)$ space. Let $a_i$ be a fixed element in $X_i$. Let $A_i = \{ x \in X = \Pi_{i\in I} X_i: x_j = a_j \text{ for some } i \neq j \}$. Then $A_i$ is a subset of $X$, and hence $(A_i, s_{A_i}, t_{A_i})$ is a subspace of $(X, s, t)$. Since $(A_i, s_{A_i}, t_{A_i})$ is $FPT_0(iii)$, so $(A_i, s_{A_i}, t_{A_i})$ is $FPT_0(iii)$. Again, $A_i$ is homeomorphic image of $X_i$. Therefore, for all $i \in A, (X_i, s_i, t_i)$ is $T_0(iii)$ . Similarly, one can prove the others.

4. Mappings in Fuzzy $T_0$ Bitopological Space

In this section, we discuss about order preserving property (see Appendix A) of the notions $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$ under one-one, onto, fuzzy open and fuzzy continuous mappings.

**Theorem 6**: Suppose $(X, s, t)$ and $(Y, s_i, t_i)$ are two fuzzy bitopological spaces and $f: X \rightarrow Y$ is bijective and fuzzy open map. If $(X, s, t)$ is $FPT_0(j)$ then $(Y, s_i, t_i)$ is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

**Proof**: Let $(X, s, t)$ is $FPT_0(iii)$ and $x_m, y_n$ be fuzzy singletons in $Y$ with $x \neq y$. Since $f$ is onto then there exist $x, y \in X$ with $f(x) = x', f(y) = y'$ and $x_m, y_n$ are fuzzy singletons in $X$ with $x \neq y$ as $f$ is one-one. Again, $(X, s, t)$ is $FPT_0(iii)$, there exists $u \in (s \cup t)$ such that $x_m q(u), y_n \cap u = 0$ or $v \in (s \cup t)$ such that $y_n q(u), x_m \cap v = 0$. Now, $x_m q(u) \Rightarrow u(x) + m > 1$ and $y_n \cap u = 0 \Rightarrow u(y) = 0$. Then $f(u)(x') = \{ sup u(x): f(x) = x' \} \Rightarrow f(u)(x) = u(x)$, for some $x$ and $f(u)(y') = \{ sup u(y): f(y) = y' \} \Rightarrow f(u)(y') = u(y)$, for some $y$. Also, since $f$ is a fuzzy open hence $f(u) \in (s_i \cup t_i)$ as $u \in (s \cup t)$. Again, $u(x) + m > 1 \Rightarrow f(u)(x') + m > 1 \Rightarrow x_m q f(u)$ and $u(y) = 0 \Rightarrow f(u)(y') = 0 \Rightarrow y_n \cap f(u) = 0$. It follows that there exists $f(u) \in (s_i \cup t_i)$ such that $x_m q f(u), y_n \cap f(u) = 0$. Thus, $(Y, s_i, t_i)$ is $FPT_0(iii)$. Similarly, one can prove the others.

**Theorem 7**: Suppose $(X, s, t)$ and $(Y, s_i, t_i)$ are two fuzzy bitopological spaces and $f: X \rightarrow Y$ is one-one and fuzzy continuous map. If $(Y, s_i, t_i)$ is $FPT_0(j)$ then $(X, s, t)$ is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

**Proof**: Let $(Y, s_i, t_i)$ is $FPT_0(iii)$ and $x_m, y_n$ be fuzzy singletons in $X$ with $x \neq y$. Then $(f(x))_m, (f(y))_n$ are fuzzy singletons in $Y$ with $f(x) \neq f(y)$ as $f$ is one-one. Also, since $(Y, s_i, t_i)$ is $T_0(iii)$, there exists $u \in (s_i \cup t_i)$ such that $(f(x))_m q u, (f(y))_n q \bar{u} \text{ or there exists } v \in (s_i \cup t_i)$ such that $(f(y))_n v, (f(x))_m \bar{v} \text{ and } v \in (s_i \cup t_i)$ such that $(f(y))_n q u, (f(x))_m \bar{v}$. Now, $(f(x))_m q u \Rightarrow u(f(x)) + m > 1 \Rightarrow f^{-1}(u(x)) + m > 1 \Rightarrow y_n q(f^{-1}(u))$ and $(f(y))_n \bar{u} \Rightarrow u(y) + n > 1 \Rightarrow f^{-1}(u(y)) + n > 1 \Rightarrow y_n q(f^{-1}(u))$. Since $f$ is fuzzy continuous and $u \in (s_i \cup t_i)$ hence $f^{-1}(u) \in (s \cup t)$ . It follows that there exists $f^{-1}(u) \in (s \cup t)$ such that $x_m q(f^{-1}(u)), y_n q(f^{-1}(u))$. Thus $(X, s, t)$ is $FPT_0(iii)$. Proof of others is similar to above.

5. Initial and Final Fuzzy $T_0$ Bitopological Space

**Definition 2**: [14] The initial fuzzy bitopological space on a set $X$ for the family of fuzzy bitopological spaces $\{ (X_i, s_i, t_i) \}_{i \in I}$ is the smallest fuzzy bitopology on $X$ making each $f_i$ fuzzy continuous. It is easily seen that it is generated by the family $\{ f_i^{-1}(u_i): u_i \in (s_i \cup t_i) \}_{i \in I}$.

**Definition 3**: [14] The final fuzzy bitopology on a set $X$ for the family of fuzzy bitopological spaces $\{ (X_i, s_i, t_i) \}_{i \in I}$ is the finest fuzzy bitopology on $X$ making each $f_i$ fuzzy continuous.

**Theorem 8**: If $\{ (X_i, s_i, t_i) \}_{i \in I}$ is a family of $FPT_0(j)$ fls and $\{ f_i: X \rightarrow (X_i, s_i, t_i) \}_{i \in I}$, a family of one-one and fuzzy continuous functions, then the initial fuzzy bitopology on $X$ for the family $\{ f_i \}_{i \in I}$ is $FPT_0(j)$, for $j = i, ii, iii, iv, v, vi, vii$.

**Proof**: We shall prove the above theorem for $j = i$ and the remaining is similar. Let $t_i$ be the initial fuzzy topologies on $X$ for the family $\{ f_i \}_{i \in I}$. Let $x_r, y_s$ be fuzzy singletons in $X$ with $x \neq y$. Then $f_i(x), f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as $f_i$ is one-
one. Since \((X_s, s, t)\) is \(\text{FPT}_0(iii)\), then for any two distinct fuzzy singletons \((f_i(x))_r\), \((f_j(y))_r\) in \(X_s\), there exists fuzzy set \(u_i \in (s_i \cup t_i)\) such that \((f_i(x))_q u_i, (f_j(y))_r \cap u_i = 0\) and there exists fuzzy set \(v_i \in (s_i \cup t_i)\) such that \((f_j(y))_r q v_i, (f_i(x))_r \cap v_i = 0\). Now, \((f_i(x))_q u_i \Rightarrow u_i(f_i(x)) + r > 1 \Rightarrow f_i^{-1}(u_i)(x) + r > 1\). This is true for every \(i \in \Lambda\). So, \(\inf f_i^{-1}(u_i)(x) + r > 1\) and \((f_j(y))_r \cup u_i = 0 \Rightarrow u_i(f_j(y)) = 0 = f_i^{-1}(u_i)(y) = 0\). This is true for every \(i \in \Lambda\).

So, \(\inf f_i^{-1}(u_i)(y) = 0\). Let \(u = \inf f_i^{-1}(u_i)\). Then \(u \in (s_i \cup t_i)\) as \(f_i\) is fuzzy continuous. So \(u(x) + r > 1\) and \(u(y) = 0\). Hence \(x \neq y\) and \(y \neq u\). Therefore, \((X_s, t)\) is \(\text{FPT}_0(ii)\).

**Theorem 9:** If \(\{X_i, s_i, t_i\}_{i \in \Lambda}\) is a family of \(\text{FPT}_0(j)\) sets and \(\{f_i: (X_s, s_i \cup t_i) \rightarrow X\}_{i \in \Lambda}\), a family of fuzzy open and bijective function, then the final fuzzy topology on \(X\) for the family \(\{f_i\}_{i \in \Lambda}\) is \(\text{FPT}_0(f)\), for \(j = i, ii, iii, iv, vi, vii\).

**Proof:** We shall prove the above theorem for \(j = ii\) and the remaining is similar. Let \(s, t\) be the final fuzzy topologies on \(X\) for the family \(\{f_i\}_{i \in \Lambda}\). Let \(x, y\) be fuzzy singletons in \(X\) with \(x \neq y\). Then \(f_i^{-1}(x), f_j^{-1}(y) \in X_i\) and \(f_i^{-1}(x) \neq f_j^{-1}(y)\) as \(f_i\) is bijective. Since \((X_s, s, t)\) is \(\text{FPT}_0(ii)\), then for any two distinct fuzzy singletons \((f_i^{-1}(x))_r, (f_j^{-1}(y))_r\) in \(X_s\), there exists fuzzy set \(u_i \in (s_i \cup t_i)\) such that \((f_i^{-1}(x))_q u_i, (f_j^{-1}(y))_r \cap u_i = 0\) or there exists fuzzy set \(v_i \in (s_i \cup t_i)\) such that \((f_j^{-1}(y))_r q v_i, (f_i^{-1}(x))_r \cap v_i = 0\). Now, \((f_j^{-1}(y))_r q u_i \Rightarrow u_i(f_j^{-1}(x)) + r > 1 \Rightarrow f_j^{-1}(u_i)(x) + r > 1\). This is true for every \(i \in \Lambda\). So, \(\inf f_j^{-1}(u_i)(x) + r > 1\) and \((f_i^{-1}(y))_r \cap u_i = 0 \Rightarrow u_i(f_i^{-1}(y)) = 0 = f_j^{-1}(u_i)(y) = 0\). This is true for every \(i \in \Lambda\). So, \(\inf f_j^{-1}(u_i)(y) = 0\). Let \(u = \inf f_j^{-1}(u_i)\). Then \(u \in (s_i \cup t_i)\) as \(f_i\) is fuzzy open. So, \(u(x) + r > 1\) and \(u(y) = 0\). Hence \(x \neq u\) and \(y \neq u\). Therefore, \((X_s, t)\) is \(\text{FPT}_0(ii)\).

6. Conclusions
We introduce and study some definitions of \(T_0\) – type separation in fuzzy bitopological space by using quasi-coincidence sense. We have shown that all of the above definitions of ours are good extension of their counterparts in [10, 11] and are finer than other such definition in [13]. Finally, we have discussed initial and final fuzzy \(T_0\) bitopological spaces according to [14].

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References
Theorem into the closed unit interval
Definition implies that each coordinate space has property
bitopological space, each having property
product fuzzy topologies of the family
A fuzzy bitopological property
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Appendix A: Basic Notions

**Definition A.1:** [1] A function $u$ from $X$ into the unit interval $I$ is called a fuzzy set in $X$. For every $x \in X$, $u(x) \in I$ is called the grade of membership of $x$ in $u$. Some authors say that $u$ is a fuzzy subset of $X$ instead of saying that $u$ is a fuzzy set in $X$.

The class of all fuzzy sets from $X$ into the closed unit interval $I$ will be denoted by $I^X$.

**Definition A.2:** [15] A fuzzy set $u$ in $X$ is called a fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$, for a certain $x \in X$ and $u(y) = 0$ for all points $y$ of $X$ except $x$. The fuzzy singleton is denoted by $x$, and $x$ is its support. The class of all fuzzy singletons in $X$ will be denoted by $S(X)$. If $u \in I^X$ and $x, y \in S(X)$, then we say that $x, y \in u$ if and only if $r \leq u(x)$.

**Definition A.3:** [16] A fuzzy set $u$ in $X$ is called a fuzzy point if and only if $u(x) = r, 0 < r < 1$, for a certain $x \in X$ and $u(y) = 0$ for all points $y$ of $X$ except $x$. The fuzzy point is denoted by $x$, and $x$ is its support.

**Definition A.4:** [9] A fuzzy singleton $x$, is said to be quasi-coincidence with $u$, denoted by $x, qu$ if and only if $u(x) + r > 1$. If $x, y$ is not quasi-coincidence with $u$, we write $x, qu$ and defined as $u(x) + r \leq 1$.

**Definition A.5:** [2] Let $f$ be a mapping from a set $X$ into a set $Y$ and $\nu$ be a fuzzy subset of $Y$. Then the inverse of $\nu$ written as $f^{-1}(\nu)$ is a fuzzy subset of $X$ defined by $f^{-1}(\nu)(x) = \nu(f(x))$, for $x \in X$.

**Definition A.6:** [17] The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $\nu \in s$, $f^{-1}(\nu) \in t$, the function $f$ is called fuzzy homeomorphic if and only if $f$ is bijective and both $f$ and $f^{-1}$ are fuzzy continuous.

**Definition A.7:** [18] The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set $u$ in $(X, t)$, $f(u)$ is open fuzzy set in $(Y, s)$.

**Definition A.8:** [19] Let $f$ be a real valued function on a topological space. If $\{x : f(x) > a\}$ is open for every real $a$, then $f$ is called lower semi continuous function.

**Definition A.9:** [19] A bitopological space $(X, S, T)$ is called pairwise-$PT_0$ ($PT_0$ in short) if for all $x, y \in X, x \neq y$, there exist $U \in S \cup T$ such that $x \in U, y \notin U$ or $x \notin U, y \in U$.

**Definition A.10:** [13] A fuzzy bitopological space $(X, t, s)$ is called $FPT_0$ if and only if for all $x, y \in X$ with $x \neq y$, there exists $u \in s \cup t$ such that $u(x) = 1, u(y) = 0$ or $u(y) = 1, u(x) = 0$.

Further, if $A \subseteq X$ and $s_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$ denotes the subspace topology on $A$ induced by $s_A, t_A$. Then $(A, s_A, t_A)$ is called subspace of $(X, s, t)$ with the underlying set $A$.

A fuzzy bitopological property $P$ is called hereditary if each subspace of a fuzzy bitopological space with property $P$, also has property $P$.

**Definition A.11:** [20] Let $(X_i, s_i, t_i) : i \in A$ is a family of fuzzy bitopological spaces. Then the space $(\Pi X_i, \Pi s_i, \Pi t_i)$ is called the product fuzzy bitopological space of the family $(X_i, s_i, t_i) : i \in A$, where $\Pi s_i, \Pi t_i$ respectively denote the usual product fuzzy topologies of the families $\{s_i : i \in A\}$ and $\{t_i : i \in A\}$ of the fuzzy topologies on $X_i$.

A fuzzy bitopological property $P$ is called projective if the product of fuzzy bitopological spaces of a family of fuzzy bitopological spaces, each having property $P$, has property $P$. A fuzzy bitopological property $P$ is called projective if for a family of fuzzy bitopological spaces $(X_i, s_i, t_i) : i \in A$, the product fuzzy bitopological space $(\Pi X_i, \Pi s_i, \Pi t_i)$ has property $P$ implies that each coordinate space has property $P$.

**Definition A.12:** [21] Let $(X, T)$ be an ordinary topological space. The set of all lower semi continuous functions from $(X, T)$ into the closed unit interval $I$ equipped with the usual topology constitute a fuzzy topology associated with $(X, T)$ and is denoted as $(X, \omega(T))$.

**Theorem A.1:** [12] A bijective mapping from an fts $(X, t)$ to an fts $(Y, s)$ preserves the value of a fuzzy singleton (fuzzy point).