Available online at https://ganitjournal.bdmathsociety.org/

GANIT: Journal of Bangladesh Mathematical Society

GANIT

GANIT *J. Bangladesh Math. Soc.* 40.2 (2020) 156–162
DOI: *https://doi.org/10.3329/ganit.v40i2.51309*

Separation Axiom (T_0) on Fuzzy Bitopological Space in Quasi-coincidence Sense

Saikh Shahjahan Miah[a,*], M R Amin[b], and Muhammad Shahjalal[c]

[a]*Department of Mathematics, Faculty of Science, Mawlana Bhashani Science and Technology University Tangail-1902, Bangladesh*
[b]*Department of Mathematics, Faculty of Science, Begum Rokeya University, Rangpur-5404, Bangladesh*
[c]*Department of Mathematics, Faculty of Science, Bangamata Sheikh Fojilatunnesa Mujib Science and Technology University, Jamalpur-2000, Bangladesh*

ABSTRACT

In this paper, we introduce two notions of T_0 separation on fuzzy bitopological space using quasi-coincidence sense and establish relations among ours and other counterparts. We show that our notions satisfy "good extension", "hereditary", "productive", "projective" properties and observe that our concepts are preserved under one-one, onto, fuzzy open and fuzzy continuous mappings. Finally, we discuss initial and final fuzzy bitopological spaces on the mentioned concepts.

Received: June 16, 2020 **Accepted:** November 23, 2020 **Published in Online:** January 15, 2021

Keywords: Fuzzy Bitopological Space; Quasi-coincidence; Fuzzy T_0 Bitopological Space; Good Extension; Initial and Final Fuzzy Bitopological Spaces.

AMS Subject Classifications 2020: *54A40, 03E72*

1. Introduction

The first concept of fuzzy sets introduced by Zadeh [1] in 1965. By using this concept Chang [2] defined fuzzy topological spaces in 1968. Separation axioms [3, 4, 5] are important parts in fuzzy topological spaces. Also many researchers have contributed various types of separation axioms [6, 7, 8] on fuzzy bitopological space which is introduced by Kandil and El-Shafee [9] in 1991. Among those axioms, T_0 type separation on fuzzy bitopological space is one and it has been introduced earlier by Abu Sufia *et al.* [10], Nouh [11], Amin *et al.* [12] and others.

The purpose of this paper is to further contribute to the development of fuzzy bitopological spaces especially on fuzzy T_0 bitopological spaces. In the present paper, fuzzy T_0 bitopological space is defined by using quasi-coincidence sense and it is showed that the good extension property is satisfied on our notions. In the next section of this paper, it is also showed that the hereditary, order preserving, productive, and projective properties hold on the new concepts. Initial and final fuzzy bitopologies are discussed also.

2. Fuzzy T_0 Bitopological Space

In this section, we discuss about our notions and findings. Some well-known properties are discussed here by using our concepts. Here X and Y always denote nonempty sets.

* Corresponding author: Saikh Shahjahan Miah, *E-mail address: skhshahjahan@gmail.com*

Definition 1: *A fuzzy bitopological space* (X, s, t) *is called*
(a) [10] $\boldsymbol{FPT_0(i)}$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exists* $u \in (s \cup t)$ *such that* $x_m \in u$, $y_n \bar{q} u$, *or there exists* $v \in (s \cup t)$ *such that* $y_n \in v$, $x_m \bar{q} v$.
(b) $\boldsymbol{FPT_0(ii)}$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exists* $u \in (s \cup t)$ *such that* $x_m q u$, $y_n \cap u = 0$ *or there exists* $v \in (s \cup t)$ *such that* $y_n q v$, $x_m \cap v = 0$.
(c) $\boldsymbol{FPT_0(iii)}$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exists* $u \in (s \cup t)$ *such that* $x_m q u$, $y_n \bar{q} u$, *or there exists* $v \in (s \cup t)$ *such that* $y_n q v$, $x_m \bar{q} v$.
(d) [10] $\boldsymbol{FPT_0(iv)}$ *if and only if for any pair of distinct fuzzy points* p, q *in X, there exists a fuzzy set* $u \in (s \cup t)$ *such that* $p \in u$, $q \cap u = 0$, *or* $q \in u$, $p \cap u = 0$.
(e) [11] $\boldsymbol{FPT_0(v)}$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exists* $u \in s$ *or* $u \in t$ *which is Q-nbd of one of the fuzzy singletons and not quasi-coincident with other.*
(f) [10] $\boldsymbol{FPT_0(vi)}$ *if and only if for any pair* $x_m, y_n \in S(X)$ *with* $x \neq y$*, there exists* $u \in (s \cup t)$ *such that* $x_m \in u$, $u \subseteq (y_n)^c$, *or* $y_n \in v$, $u \subseteq (x_m)^c$.
(g) [13] $\boldsymbol{FPT_0(vii)}$ *if and only if for all* $x, y \in X$ *with* $x \neq y$*, there exists* $u \in (s \cup t)$ *such that* $u(x) = 1, u(y) = 0$ *or* $u(y) = 1, u(x) = 0$.

Example 1: Let $X = \{x, y\}, u \in I^X$ with $u(x) = 1, u(y) = 0$ and t be the fuzzy topology on X generated by $\{0, u, 1\}$ and s be the fuzzy topology on X generated by {constants}. Also, let $x_m, y_n \in S(X)$ with $x \neq y$. Then $u(x) + m > 1$ and $u(y) + s \leq 1$ for $m, n \in (0,1]$. Thus, $x_m q u$, $y_n \bar{q} u$ that is x_m is quasi-coincidence with u, y_n is not quasi-coincidence with u. Hence (X, s, t) is $FPT_0(iii)$ as $u \in (s \cup t)$. Also, as $u(y) = 0, y_n \cap u = 0$. Therefore, (X, s, t) is $FPT_0(ii)$.

Theorem 1: *Let* (X, s, t) *be a fuzzy bitopological space and* $(X, s \cup t)$ *be a fuzzy topological space. Then* (X, s, t) *is* FPT_0 *if and only if* $(X, s \cup t)$ *is* FT_0.

Proof: Let (X, s, t) is FPT_0. Since $s \subseteq (s \cup t)$ and $t \subseteq (s \cup t)$, it follows that $(X, s \cup t)$ is FT_0.
Conversely, let $(X, s \cup t)$ is FT_0 and x_m, y_n are fuzzy singletons in X with $x \neq y$, then $\exists\, u \in (s \cup t)$ such that $x_m \in u$ and $y_n \bar{q} u$ or $\exists\, v \in (s \cup t)$ such that $y_n \in v$ and $x_m \bar{q} v$. Hence (X, s, t) is FPT_0.

Theorem 2: *If the fuzzy topological space* (X, s) *and* (X, t) *are both* $FT_0(j)$*, then their corresponding fuzzy bitopological space* (X, s, t) *is* $FPT_0(j)$*, for* $j = i, iii$*. But the converse is not true in general.*

Proof: Let (X, s) and (X, t) are both $FT_0(j)$. Then their corresponding fuzzy bitopological space (X, s, t) is $FPT_0(j)$, for $j = i, iii$ as $s \subseteq (s \cup t)$ and $t \subseteq (s \cup t)$. To prove (X, s, t) is $FPT_0(j)$ does not imply (X, s) and (X, t) are both $FT_0(j)$, for $j = i, iii$, we will give a counter example.

Counter example 1: Let $X = \{x, y\}, u \in I^X$ and t be the fuzzy topology on X generated by $\{u\} \cup \{constants\}$, with $u(x) = 1, u(y) = 0$. Also, let s be the fuzzy topology on X generated by $\{constants\}$. Then for any $0 < m \leq 1$ and $0 < n \leq 1$, $u(x) + m > 1$ and $u(y) + n \leq 1$. Thus $x_m q u, y_n \bar{q} u$. As $u \subseteq (s \cup t)$, (X, s, t) is $FPT_0(j)$ but (X, s) is not $FT_0(j)$, for $j = i, iii$.

Theorem 3: *For a fuzzy bitopological space* (X, s, t) *the following implications are true:*

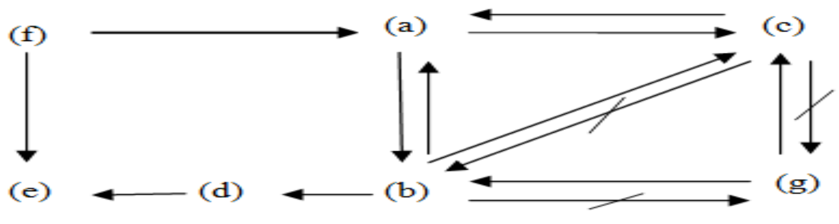

Figure 2.1: Implications of notions

where (a), (b), (c), (d), (e), (f) and (g) are used, for convenience, instead of $FPT_0(i)$, $FPT_0(ii)$, $FPT_0(iii)$, $FPT_0(iv)$, $FPT_0(v)$, $FPT_0(vi)$ *and* $FPT_0(vii)$ *respectively.*

Proof: $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b})$**:** Let (X, s, t) be $FPT_0(i)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Also let $r > 1 - m$ for $0 < m < 1$. Since (X, s, t) is $FPT_0(i)$, there exists $u \in (s \cup t)$ such that $x_r \in u, y_1 \bar{q}\, u$.
Now, $x_r \in u \Rightarrow u(x) \geq r > 1 - m \Rightarrow u(x) + m > 1 \Rightarrow x_m q u$ and
$y_1 \bar{q}\, u \Rightarrow u(y) + 1 \leq 1 \Rightarrow u(y) \leq 1 - 1 = 0 \Rightarrow u(y) = 0 \Rightarrow y_n \cap u = 0$.
It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_m q u$, $y_n \cap u = 0$. Thus (X, s, t) is $FPT_0(ii)$.

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{d})$**:** Let x_m, y_n be distinct fuzzy singletons in X and $0 < r < 1$ with $r \leq 1 - m$. Since (X, s, t) is $FPT_0(ii)$, there exists $u \in s$ such that $x_r qu$ and $y_n \cap u = 0$ or there exists $v \in t$ such that $y_n qv$ and $x_r \cap v = 0$.
Now, $x_r qu \Rightarrow u(x) + r > 1 \Rightarrow u(x) > 1 - r \geq m \Rightarrow u(x) \geq m \Rightarrow x_m \in u$
It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_m \in u$, $y_n \cap u = 0$. Thus (X, s, t) is $FPT_0(iv)$.
$(\boldsymbol{d}) \Rightarrow (\boldsymbol{e})$**:** Let x_m, y_n be distinct fuzzy singletons in X and $0 < m_1 < 1$, $0 < n_1 < 1$ with $m_1 > 1 - m$ and $n_1 > 1 - n$. Since (X, s, t) is $FPT_0(iv)$ and x_{m_1}, y_{n_1} are distinct fuzzy points, there exists $u \in (s \cup t)$ such that $x_{m_1} \in u$ and $y_{n_1} \cap u = 0$.
Now, $x_{m_1} \in u \Rightarrow u(x) \geq m_1 > 1 - m \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and
$y_{n_1} \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$
It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_m qu$, $y_n \bar{q} u$. Thus (X, s, t) is $FPT_0(v)$.
$(\boldsymbol{e}) \Rightarrow (\boldsymbol{f})$**:** Let x_m, y_n be distinct fuzzy singletons in X and $0 < r < 1$ with $r \leq 1 - m$. Since (X, s, t) is $FPT_0(v)$, there exists $u \in (s \cup t)$ such that $x_r qu$ and $y_n \bar{q} u$.
Now, $x_r qu \Rightarrow u(x) + r > 1 \Rightarrow u(x) > 1 - r \geq m \Rightarrow u(x) \geq m \Rightarrow x_m \in u$ and
$y_n \bar{q} u \Rightarrow u(y) + n \leq 1 \Rightarrow n \leq 1 - u(y) \Rightarrow u \subseteq (y_n)^c$
It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_m \in u$, $u \subseteq (y_n)^c$. Therefore, (X, s, t) is $FPT_0(vi)$.
$(\boldsymbol{f}) \Rightarrow (\boldsymbol{a})$**:** Let x_m, y_n be distinct fuzzy singletons in X. Since (X, s, t) is $FPT_0(vi)$, there exists $u \in (s \cup t)$ such that $x_m \in u$ and $u \subseteq (y_n)^c$.
Now, $u \subseteq (y_n)^c \Rightarrow n \leq 1 - u(y) \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$
It follows that for any fuzzy singletons x_m, y_n in X with $x \neq y$ there exists $u \in (s \cup t)$ such that $x_m \in u$, $y_n \bar{q} u$. Therefore, (X, s, t) is $FPT_0(i)$.
$(\boldsymbol{b}) \Rightarrow (\boldsymbol{c})$**:** Let (X, s, t) be $FPT_0(ii)$and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_0(ii)$, there exists $u \in s$ such that $x_m qu$, $y_n \cap u = 0$ or there exists $v \in t$ such that $y_n qv$, $x_m \cap v = 0$ i.e. we can say that $u \in (s \cup t)$ as $u \in s$ or $v \in s \cup t$ as $v \in t$. To prove (X, s, t) is $FPT_0(iii)$, it is only needed to prove that $y_n \bar{q}\, u$ or $x_m \bar{q}\, v$.
Now, $y_n \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q}\, u$. Thus (X, s, t) is $FPT_0(ii)$.
To show $(c) \nRightarrow (b)$, we shall give a counter example.

Counter example 2: Let $X = \{x, y\}$ and $u, v \in I^X$ be given by $u(x) = 1$, $u(y) = 0.1$, $v(y) = 1$, $v(x) = 0.1$. Let us consider the fuzzy topology s on X generated by $\{0, u, 1\}$ and the fuzzy topology t on X generated by $\{0, v, 1\}$.
For $0 < m \leq 1, 0 < n < 0.9$, $u(x) + m > 1 \Rightarrow x_m qu$ and $u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Thus (X, s, t) is $FPT_0(iii)$ as $u \in (s \cup t)$. But $u(y) \neq 0 \Rightarrow y_n \cap u \neq 0$. Also, $v(x) \neq 0 \Rightarrow x_m \cap v \neq 0$. Thus (X, s, t) is not $FPT_0(ii)$.
$(\boldsymbol{a}) \Rightarrow (\boldsymbol{c})$**:** As $(d) \Rightarrow (e)$ we can say that $(a) \Rightarrow (c)$.
$(\boldsymbol{g}) \Rightarrow (\boldsymbol{c})$**:** Let (X, s, t) be $FPT_0(vii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_0(vii)$, there exists a fuzzy set $u \in (s \cup t)$ such that $u(x) = 1, u(y) = 0$ or $u(x) = 0, u(y) = 1$. To prove (X, s, t) is $FPT_0(iii)$, it is only needed to prove that $x_m qu$, $y_n \bar{q}\, u$.
Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m q\, u$ and $u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Thus (X, s, t) is $FPT_0(iii)$.
To show $(c) \nRightarrow (g)$, we shall give a counter example.

Counter example 3: Let $X = \{x, y\}$ and $u \in I^X$ be given by $u(x) = 1 - \gamma$, $u(y) = 0$, where $\gamma = \frac{m}{2}$ for $m \in (0,1]$. Let the fuzzy topology s on X generated by $\{0, u, 1\}$ and the fuzzy topology t on X generated by {constants}.
Now, $u(x) = 1 - \gamma \Rightarrow u(x) = 1 - \frac{m}{2} \Rightarrow u(x) + \frac{m}{2} = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and
$u(y) = 0 \Rightarrow u(y) + n \leq 1 \Rightarrow y_n \bar{q} u$. Thus (X, s, t) is $FPT_0(iii)$ as $u \in (s \cup t)$. But $u(x) \neq 1$. Thus (X, s, t) is not $FPT_0(vii)$.
$(\boldsymbol{c}) \Rightarrow (\boldsymbol{a})$**:** As $(b) \Rightarrow (d)$ we can say that $(c) \Rightarrow (a)$.
$(\boldsymbol{g}) \Rightarrow (\boldsymbol{b})$**:** Let (X, s, t) be $FPT_0(vii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Since (X, s, t) is $FPT_0(vii)$, there exists a fuzzy set $u \in (s \cup t)$ such that $u(x) = 1, u(y) = 0$ or $u(x) = 0, u(y) = 1$. To prove (X, s, t) is $FPT_0(iii)$, it is only needed to prove that $x_m qu$, $y_n \cap u = 0$.
Now, $u(x) = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m q\, u$ and $u(y) = 0 \Rightarrow y_n \cap u = 0$. Thus (X, s, t) is $FPT_0(ii)$.
To show $(b) \nRightarrow (g)$, we shall give a counter example.

Counter example 4: Let $X = \{x, y\}$ and $u \in I^X$ be given by $u(x) = 1 - \gamma$, $u(y) = 0$, where $\gamma = \frac{m}{2}$ for $m \in (0,1]$. Let the fuzzy topology s on X generated by $\{0, u, 1\}$ and the fuzzy topology t on X generated by {constants}.
Now, $u(x) = 1 - \gamma \Rightarrow u(x) = 1 - \frac{m}{2} \Rightarrow u(x) + \frac{m}{2} = 1 \Rightarrow u(x) + m > 1 \Rightarrow x_m qu$ and
$u(y) = 0 \Rightarrow y_n \cap u = 0$. Thus (X, s, t) is $FPT_0(ii)$ as $u \in (s \cup t)$. But $u(x) \neq 1$. Thus (X, s, t) is not $FPT_0(vii)$.
$(\boldsymbol{b}) \Rightarrow (\boldsymbol{a})$**:** As $(b) \Rightarrow (d)$ and $(d) \Rightarrow (e)$ we can say that $(b) \Rightarrow (a)$. Thus proof is complete.

Theorem 4: *Let (X,S,T) be a bitopological space. Then (X,S,T) is PT_0 if and only if $(X,\omega(S),\omega(T))$ is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.*

Proof: Let (X,S,T) be a PT_0 topological space. We shall prove that $(X,\omega(S),\omega(T))$ is $FPT_0(ii)$. Let x,y in X with $x \neq y$. Since (X,S,T) be a PT_0 topological space hence there exists $U \in (S \cup T)$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Suppose that $x \in U, y \notin U$. From the definition of lower semi continuous function, $1_U \in (\omega(S) \cup \omega(T))$. i.e. $1_U \in \omega(S)$ or $1_U \in \omega(T)$. Then

$$1_U(x) = 1 \Rightarrow 1_U(x) + m > 1 \Rightarrow x_m q 1_U$$

And

$$1_U(y) = 0 \Rightarrow y_n \cap 1_U = 0$$

It follows that there exists $1_U \in \omega(S)$ or $1_U \in \omega(T)$ such that $x_m q 1_U$, $y_n \cap 1_U = 0$. Hence $(X,\omega(S),\omega(T))$ is $FPT_0(ii)$.
Conversely, let $(X,\omega(S),\omega(T))$ is $FPT_0(ii)$. It is required to prove that (X,S,T) be a PT_0 topological space. Let x,y in X with $x \neq y$. Since $(X,\omega(S),\omega(T))$ is $FPT_0(ii)$, we have for any fuzzy singletons x_m, y_n in X, there exists $u \in \omega(S)$ such that $x_m q u$, $y_n \cap u = 0$. or $v \in \omega(T)$ such that $y_n q v$, $x_m \cap v = 0$.
Now, $x_m q u \Rightarrow u(x) + m > 1 \Rightarrow u(x) > 1 - m = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$
And $y_n \cap u = 0 \Rightarrow u(y) = 0$

$$\Rightarrow u(y) + n \leq 1 \Rightarrow u(y) \leq 1 - n = \alpha$$
$$\Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$$

Also, $u^{-1}(\alpha, 1] \in S$. It follows that there exists $u^{-1}(\alpha, 1] \in S$ such that $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$. Hence (X,S,T) be a PT_0 topological space.
Now, we shall prove the good extension property of $FPT_0(iii)$.
Let (X,S,T) be a PT_0 topological space. We shall prove that $(X,\omega(S),\omega(T))$ is $FPT_0(iii)$. Let x,y in X with $x \neq y$. Since (X,S,T) be a PT_0 topological space hence there exists $U \in (S \cup T)$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Suppose that $x \in U, y \notin U$. From the definition of lower semi continuous function, $1_U \in (\omega(S) \cup \omega(T))$. Then

$$1_U(x) = 1 \Rightarrow 1_U(x) + m > 1 \Rightarrow x_m q 1_U$$

And

$$1_U(y) = 0 \Rightarrow 1_U(y) + n \leq 1 \Rightarrow y_n \bar{q} 1_U$$

It follows that there exists $1_U \in (\omega(S) \cup \omega(T))$ such that $x_m q 1_U$, $y_n \bar{q} 1_U$. Hence $(X,\omega(S),\omega(T))$ is $FPT_0(iii)$.
Conversely, let $(X,\omega(S),\omega(T))$ is $FPT_0(iii)$. It is required to prove that (X,S,T) be a PT_0 topological space. Let x,y in X with $x \neq y$. Since $(X,\omega(S),\omega(T))$ is $FPT_0(iii)$, for any fuzzy singletons x_m, y_n in X, there exists $u \in (\omega(S) \cup \omega(T))$ such that $x_m q u$, $y_n \bar{q} u$. Now, $x_m q u \Rightarrow u(x) + m > 1 \Rightarrow u(x) > 1 - m = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$
And $y_n \bar{q} u \Rightarrow u(y) + n \leq 1 \Rightarrow u(y) \leq 1 - n = \alpha$

$$\Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$$

Also, $u^{-1}(\alpha, 1] \in (S \cup T)$. It follows that there exists $u^{-1}(\alpha, 1] \in (S \cup T)$ such that $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$. Hence (X,S,T) be a PT_0 topological space.
Proof for $j = i, iv, v, vi, vii$ is similar to above.

3. Hereditary, Productive and Projective properties

In this section, we describe the hereditary, productive and projective properties (see Appendix A) on our given concepts $FPT_0(j)$, where $j = ii, iii$. The first theorem is on hereditary property and the second one is on productive and projective properties.

Theorem 4: *If (X,s,t) be a fuzzy bitopological space and $A \subseteq X$, $s_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$ and (X,s,t) is $FPT_0(j)$ then (A, s_A, t_A) is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.*

Proof: We shall prove this theorem for $j = ii$ and remaining is similar.
Let (X,s,t) is $FPT_0(ii)$ and x_m, y_n are fuzzy singletons in A with $x \neq y$. Since $A \subseteq X$, x_m, y_n are also fuzzy singletons in X. Also since (X,s,t) is $FPT_0(ii)$, there exists $u \in (s \cup t)$ such that $x_m q u$ and $y_n \cap u = 0$ or $\exists\ v \in (s \cup t)$ such that $y_n q v$ and $x_m \cap v = 0$. For $A \subseteq X$, we have $u/A \in (s_A \cup t_A)$.
Now, $x_m q u \Rightarrow u(x) + m > 1,\ x \in X \Rightarrow u/A(x) + m > 1, x \in A \subseteq X \Rightarrow x_m q u/A$
And $y_n \cap u = 0 \Rightarrow u(y) = 0,\ y \in X \Rightarrow u/A(y) = 0,\ y \in A \subseteq X \Rightarrow y_n \cap u/A = 0$
Therefore, (A, s_A, t_A) is $FPT_0(ii)$.

Theorem 5: *If $\{(X_i, s_i, t_i) : i \in \Lambda\}$ is a family of fuzzy bitopological spaces then the product fuzzy bitopological spaces $(\Pi X_i, \Pi s_i, \Pi t_i) = (X,s,t)$ is $FPT_0(j)$ if and only if each coordinate space (X_i, s_i, t_i) is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.*

Proof: Let for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_0(iii)$ space. Let x_m, y_n be fuzzy singletons in X with $x \neq y$. Then $(x_i)_m, (y_i)_n$ are fuzzy singletons with $x_i \neq y_i$ for some $i \in \Lambda$. Since (X_i, s_i, t_i) is $FPT_0(iii)$, there exists $u_i \in (s_i \cup t_i)$ such that $(x_i)_m q u_i, (y_i)_n \bar{q} u_i$ or $v_i \in (s_i \cup t_i)$ such that $(y_i)_n q v_i, (x_i)_m \bar{q} v_i$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Now, $(x_i)_m q u_i \Rightarrow u_i(x_i) + m > 1 \Rightarrow u_i(\pi_i(x)) + m > 1 \Rightarrow (u_i \circ \pi_i)(x) + m > 1 \Rightarrow x_m q(u_i \circ \pi_i)$ and $(y_i)_n \bar{q} u_i \Rightarrow u_i(y_i) + n \leq 1 \Rightarrow u_i(\pi_i(y)) + n \leq 1 \Rightarrow (u_i \circ \pi_i)(y) + n \leq 1 \Rightarrow y_n \bar{q}\ (u_i \circ \pi_i) = 0$. It follows that there exists $(u_i \circ \pi_i) \in (s_i \cup t_i)$ such that $x_m q(u_i \circ \pi_i), y_n \bar{q}\ (u_i \circ \pi_i)$. Hence, (X, s, t) is $FPT_0(iii)$.
Conversely, let the product fuzzy bitopological space (X, s, t) is $FPT_0(iii)$. It is required to prove that for all $i \in \Lambda, (X_i, s_i, t_i)$ is $FPT_0(iii)$ space. Let a_i be a fixed element in X_i. Let $A_i = \{x \in X = \Pi_{i \in \Lambda} \in X_i : x_j = a_j$ for some $i \neq j\}$. Then A_i is a subset of X, and hence (A_i, s_{A_i}, t_{A_i}) is a subspace of (X, s, t). Since (X, s, t) is $FPT_0(iii)$, so(A_i, s_{A_i}, t_{A_i}) is $FPT_0(iii)$. Again, A_i is homeomorphic image of X_i. Therefore, for all $i \in \Lambda$, (X_i, s_i, t_i) is $T_0(iii)$. Similarly, one can prove the others.

4. Mappings in Fuzzy T_0 Bitopological Space

In this section, we discuss about order preserving property (see Appendix A) of the notions $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$ under one-one, onto, fuzzy open and fuzzy continuous mappings.

Theorem 6: *Suppose (X, s, t) and (Y, s_1, t_1) are two fuzzy bitopological spaces and $f: X \to Y$ is bijective and fuzzy open map. If (X, s, t) is $FPT_0(j)$ then (Y, s_1, t_1) is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.*

Proof: Let (X, s, t) is $FPT_0(ii)$ and x'_m, y'_n be fuzzy singletons in Y with $x' \neq y'$. Since f is onto then there exist $x, y \in X$ with $f(x) = x', f(y) = y'$ and x_m, y_n are fuzzy singletons in X with $x \neq y$ as f is one-one. Again, (X, s, t) is $FPT_0(ii)$, there exists $u \in (s \cup t)$ such that $x_m q u$, $y_n \cap u = 0$ or $v \in (s \cup t)$ such that $y_n q v$, $x_m \cap v = 0$.
Now, $x_m q u \Rightarrow u(x) + m > 1$ and $y_n \cap u = 0 \Rightarrow u(y) = 0$.
Now, $f(u)(x') = \{sup\, u(x) : f(x) = x'\} \Rightarrow f(u)(x') = u(x)$, for some x and $f(u)(y') = \{sup\, u(y) : f(y) = y'\} \Rightarrow f(u)(y') = u(y)$, for some y.
Also, since f is a fuzzy open hence $f(u) \in (s_1 \cup t_1)$ $as\ u \in (s \cup t)$.
Again, $u(x) + m > 1 \Rightarrow (f(u))(x') + m > 1 \Rightarrow x'_m q\, f(u)$ and
$u(y) = 0 \Rightarrow f(u)(y') = 0 \Rightarrow y'_n \cap f\ (u) = 0$. It follows that there exists $f(u) \in (s_1 \cup t_1)$ such that $x'_m q\, f(u)$, $y'_n \cap f\ (u) = 0$. Thus, (Y, s_1, t_1) is $FPT_0(ii)$. Similarly, one can prove the others.

Theorem 7: *Suppose (X, s, t) and (Y, s_1, t_1) are two fuzzy bitopological spaces and $f: X \to Y$ is one-one and fuzzy continuous map. If (Y, s_1, t_1) is $FPT_0(j)$ then (X, s, t) is $FPT_0(j)$, where $j = i, ii, iii, iv, v, vi, vii$.*

Proof: Let (Y, s_1, t_1) is $FPT_0(iii)$ and x_m, y_n be fuzzy singletons in X with $x \neq y$. Then $(f(x))_m$, $(f(y))_n$ are fuzzy singletons in Y with $f(x) \neq f(y)$ as f is one-one. Also, since (Y, s_1, t_1) is $T_0(iii)$, there exists $u \in (s_1 \cup t_1)$ such that $(f(x))_m\, q\, u$, $(f(y))_n \bar{q}\, u$ or there exists $v \in (s_1 \cup t_1)$ such that $(f(y))_n\, q\, v$, $(f(x))_m \bar{q}\, v$.
Now, $(f(x))_m\, q\, u \Rightarrow u(f(x)) + m > 1 \Rightarrow f^{-1}(u(x)) + m > 1 \Rightarrow (f^{-1}(u))(x) + m > 1 \Rightarrow x_m\, q(f^{-1}(u))$ and $(f(y))_n \bar{q}\, u \Rightarrow u(f(y)) + n \leq 1 \Rightarrow f^{-1}(u(y)) + n \leq 1 \Rightarrow (f^{-1}(u))(y) + n \leq 1 \Rightarrow y_n \bar{q}(f^{-1}(u))$. Since f is fuzzy continuous and $u \in (s_1 \cup t_1)$ hence $f^{-1}(u) \in (s \cup t)$. It follows that there exists $f^{-1}(u) \in (s \cup t)$ such that $x_m\, q(f^{-1}(u)), y_n \bar{q}(f^{-1}(u))$. Thus (X, s, t) is $FPT_0(iii)$. Proof of others is similar to above.

5. Initial and Final Fuzzy T_0 Bitopological Space

Definition 2: [14] The initial fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i \in \Lambda}$ and the family of functions $\{f_i: X \longrightarrow (X_i, s_i \cup t_i)\}_{i \in \Lambda}$ is the smallest fuzzy bitopology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i):\ u_i \in (s_i \cup t_i)\}_{i \in \Lambda}$.

Definition 3: [14] The final fuzzy bitopology on a set X for the family of fuzzy bitopological spaces $\{(X_i, s_i, t_i)\}_{i \in \Lambda}$ and the family of functions $\{f_i: (X_i, s_i \cup t_i) \longrightarrow X\}_{i \in \Lambda}$ is the finest fuzzy bitopology on X making each f_i fuzzy continuous.

Theorem 8: *If $\{(X_i, s_i, t_i)\}_{i \in \Lambda}$ is a family of $FPT_0(j)$ fts and $\{f_i: X \longrightarrow (X_i, s_i \cup t_i)\}_{i \in \Lambda}$, a family of one-one and fuzzy continuous functions, then the initial fuzzy bitopology on X for the family $\{f_i\}_{i \in \Lambda}$ is $FPT_0(j)$, for $j = i, ii, iii, iv, vi, vii$.*

Proof: We shall prove the above theorem for $j = ii$ and the remaining is similar. Let t, s be the initial fuzzy topologies on X for the family $\{f_i\}_{i \in \Lambda}$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i(x), f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one-

one. Scince (X_i, s_i, t_i) is $FPT_0(ii)$, then for any two distinct fuzzy singletons $(f_i(x))_r$, $(f_i(y))_s$ in X_i, there exists fuzzy set $u_i \in (s_i \cup t_i)$ such that $(f_i(x))_r q\, u_i$, $(f_i(y))_s \cap u_i = 0$ and or there exists fuzzy set $v_i \in (s_i \cup t_i)$ such that $(f_i(y))_s q\, v_i$, $(f_i(x))_r \cap v_i = 0$.
Now, $\left(f_i(x)\right)_r q\, u_i \Rightarrow u_i\left(f_i(x)\right) + r > 1 \Rightarrow f_i^{-1}(u_i)(x) + r > 1$. This is true for every $i \epsilon \Lambda$. So, $inf\ f_i^{-1}(u_i)(x) + r > 1$ and $\left(f_i(y)\right)_s \cap u_i = 0 \Rightarrow u_i\left(f_i(y)\right) = 0 \Rightarrow f_i^{-1}(u_i)(y) = 0$. This is true for every $i \epsilon \Lambda$.
So, $\inf f_i^{-1}(u_i)(y) = 0$. Let $u = inf\ f_i^{-1}(u_i)$. Then $u \in (s_i \cup t_i)$ as f_i is fuzzy continuous. So $u(x) + r > 1$ and $u(y) = 0$. Hence $x_r qu$ and $y_s \cap u = 0$. Therefore, (X, s, t) is $FPT_0(ii)$.

Theorem 9: *If* $\{(X_i, s_i, t_i)\}_{i\epsilon\Lambda}$ *is a family of* $FPT_0(j)$ *fts and* $\{f_i: (X_i, s_i \cup t_i) \longrightarrow X\}_{i\epsilon\Lambda}$, *a family of fuzzy open and bijective function, then the final fuzzy topology on X for the family* $\{f_i\}_{i\epsilon\Lambda}$ *is* $FPT_0(j)$, *for* $j = i, ii, iii, iv, v, vi, vii$.

Proof: We shall prove the above theorem for $j = ii$ and the remaining is similar. Let s, t be the final fuzzy topologies on X for the family $\{f_i\}_{i\epsilon\Lambda}$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i^{-1}(x)$, $f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Scince (X_i, s_i, t_i) is $FPT_0(ii)$, then for any two distinct fuzzy singletons $(f_i^{-1}(x))_r$, $(f_i^{-1}(y))_s$ in X_i, there exists fuzzy set $u_i \in (s_i \cup t_i)$ such that $(f_i^{-1}(x))_r q\, u_i$, $(f_i^{-1}(y))_s \cap u_i = 0$ or there exists fuzzy set $v_i \in (s_i \cup t_i)$ such that $(f_i^{-1}(y))_s q\, v_i$, $(f_i^{-1}(x))_r \cap v_i = 0$.
Now, $(f_i^{-1}(x))_r q\, u_i \Rightarrow u_i\left(f_i^{-1}(x)\right) + r > 1 \Rightarrow f_i(u_i)(x) + r > 1$. This is true for every $i \in \Lambda$. So, $inf\ f_i(u_i)(x) + r > 1$ and $(f_i^{-1}(y))_s \cap u_i = 0 \Rightarrow u_i\left(f_i^{-1}(y)\right) = 0 \Rightarrow f_i(u_i)(y) = 0$. This is true for every $i \in \Lambda$. So, $inf\ f_i(u_i)(y) = 0$. Let $u = inf\ f_i(u_i)$. Then $u \in (s_i \cup t_i)$ as f_i is fuzzy open. So, $u(x) + r > 1$ and $u(y) = 0$. Hence $x_r qu$ and $y_s \cap u = 0$. Therefore, (X, s, t) is $FPT_0(ii)$.

6. Conclusions

We introduce and study some definitions of T_0 −type separation in fuzzy bitopological space by using quasi-coincidence sense. We have shown that all of proposed definitions of ours are good extension of their counterparts in [10, 11] and are finer than other such definition in [13]. Finally, we have discussed initial and final fuzzy T_0 bitopological spaces according to [14].

Acknowledgements

We would like to express our special thanks to the respected reviewers for their valuable comments and suggestions which help us to develop the quality more of this work. Also, we are thankful to the editorial board of the journal for their unrestricted supports.

References

[1] Zadeh, L. A., Fuzzy sets, *Information and Control* 8: 338-353 (1965).
[2] Chang, C. L., Fuzzy topological spaces, *J. Math. Anal. Appl*. 24: 182-192 (1968).
[3] Miah, S. S., Amin, M. R., Mappings in fuzzy Hausdorff spaces in quasi-coincidence sense, *J. Bangladesh Acad. Sci*. 41(1): 47-56 (2017).
[4] Miah, S. S., Amin, M. R., Rashid, H., T_1 −type separation on fuzzy topological spaces in quasi- coincidence sense, *J. Mech. Cont. & Math. Sci*. 12(1): 45-56 (2017).
[5] Miah, S. S., Amin, M. R., Rana, S., Fuzzy normal topological space in quasi-coincidence sense, *J. Bangladesh Acad. Sci*. 42(2): 201-205 (2017).
[6] Miah, S. S., Amin, M. R., Certain properties in fuzzy R_0 topological spaces in quasi-coincidence sense, *Ann. Pure Appl. Math.* 14(1): 125-131 (2017).
[7] Miah, S. S., Amin, M. R., Jahan, M., Mappings on fuzzy T_0 topological spaces in quasi-coincidence sense, *J. Math. Comput. Sci.* 7(5): 883-894 (2017).
[8] Miah, S. S., Jahan, M., Amin, M. R., Separation axiom (Hausdorff) on fuzzy bitopological space in quasi-coincidence sense, *J. Sci. Eng. Res*., 6(4): 167-174 (2017).
[9] Kandil, El-Shafee, Separation axioms for fuzzy bitopological spaces, *J. Inst. Math. Comput. Sci*. 4(3): 373-383 (1991).
[10] Abu Sufiya, A. S., Fora, A. A., Warner, M. W., Fuzzy separation axioms and fuzzy continuity in fuzzy bitopological spaces, *Fuzzy Sets and Systems* 62: 367-373 (1994).
[11] Nouh, A. A., On Separation axioms in fuzzy bitopological spaces, *Fuzzy Sets and Systems* 80: 225-236 (1996).
[12] Amin, M. R., Ali, D. M., Hossain, M. S., On T_0 fuzzy bitopological spaces, *Journal of Bangladesh Academy of Sciences* 32(2): 209-217 (2014).
[13] Srivastava, M., Srivastava, R., On fuzzy pairwise-T_0 and fuzzy pairwise T_1 bitopological space, *Indian J. Pure Appl. Math*. 32(3): 387-396 (2001).

[14] Miah, S. S., Amin, M. R., Hoque, M. F., Separation axioms (T_1) on fuzzy bitopological space in quasi-coincidence sense, *International J. Math. Combin*. 3(2019): 43-53 (2019).

[15] Ming, P. P., Liu Ying Ming, Fuzzy topology I. neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl*. 76: 571-599 (1980).

[16] Wong, C. K., Fuzzy points and local properties of fuzzy topology, *J. Math. Anal. Appl*. 46: 316-328 (1974).

[17] Ming, P. P., Liu Ying Ming, Fuzzy topology II. product and quotient spaces, *J. Math. Anal. Appl*. 77: 20-37 (1980).

[18] Malghan, S. R., Benchalli, S. S., On open maps, closed maps and local compactness in fuzzy topological spaces, *J. Math. Anal. Appl.* 99(2): 338-349 (1984).

[19] Rudin, W., Real and Complex Analysis, Copyright 1966, by McGrawHill Inc (1996).

[20] Wong, C. K., Fuzzy topology: product and quotient theorem, *J. Math. Anal. Appl.* 45: 512-521 (1974).

[21] Lowen, R., Fuzzy topological spaces and fuzzy compactness, *J. Math. Anal. Appl*. 56: 621-633 (1976).

Appendix A: Basic Notions

Definition A.1: [1] A function *u* from *X* into the unit interval *I* is called a fuzzy set in *X*. For every $x \in X$, $u(x) \in I$ is called the grade of membership of *x* in *u*. Some authors say that *u* is a fuzzy subset of *X* instead of saying that *u* is a fuzzy set in *X*. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.

Definition A.2: [15] A fuzzy set u in X is called a fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$, for a certain $x \in X$ and $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in *X* will be denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$.

Definition A.3: [16] A fuzzy set *u* in *X* is called a fuzzy point if and only if $u(x) = r, 0 < r < 1$, for a certain $x \in X$ and $u(y) = 0$ for all points *y* of *X* except *x*. The fuzzy point is denoted by x_r and *x* is its support.

Definition A.4: [9] A fuzzy singleton x_r is said to be quasi-coincidence with *u*, denoted by $x_r q u$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincidence with *u*, we write $x_r \bar{q} u$ and defined as $u(x) + r \leq 1$.

Definition A.5: [2] Let *f* be a mapping from a set *X* into a set *Y* and *v* be a fuzzy subset of *Y* . Then the inverse of *v* written as $f^{-1}(v)$ is a fuzzy subset of *X* defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

Definition A.6: [17] The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function *f* is called fuzzy homeomorphic if and only if *f* is bijective and both *f* and f^{-1} are fuzzy continuous.

Definition A.7: [18] The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set *u* in $(X, t), f(u)$ is open fuzzy set in (*Y, s*).

Definition A.8: [19] Let *f* be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real *α*, then *f* is called lower semi continuous function.

Definition A.9: [19] A bitopological space (X, S, T) is called pairwise-T_0 (PT_0 in short) if for all $x, y \in X, x \neq y$, there exist $U \in S \cup T$ such that $x \in U, y \notin U$ or $x \notin U, y \in U$.

Definition A.10: [13] A fuzzy bitopological space (X, s, t) is called FPT_0 if and only if for all $x, y \in X$ with $x \neq y$, there exists $u \in s \cup t$ such that $u(x) = 1, u(y) = 0$ or $u(y) = 1, u(x) = 0$.

Further, if $A \subseteq X$ and $s_A = \{u/A : u \in s\}, t_A = \{v/A : v \in t\}$ denotes the subspace topology on A induced by s_A, t_A. Then (A, s_A, t_A) is called subspace of (X, s, t) with the underlying set A.
A fuzzy bitopological property P is called hereditary if each subspace of a fuzzy bitopological space with property P, also has property P.

Definition A.11: [20] Let $\{(X_i, s_i, t_i): i \in \Lambda\}$ is a family of fuzzy bitopological spaces. Then the space $(\Pi X_i, \Pi s_i, \Pi t_i)$ is called the product fuzzy bitopological space of the family $\{(X_i, s_i, t_i): i \in \Lambda\}$, where $\Pi s_i, \Pi t_i$ respectively denote the usual product fuzzy topologies of the families $\{\Pi s_i: i \in \Lambda\}$ and $\{\Pi t_i: i \in \Lambda\}$ of the fuzzy topologies on X.
A fuzzy bitopological property P is called productive if the product of fuzzy bitopological spaces of a family of fuzzy bitopological space, each having property P, has property P. A fuzzy bitopological property P is called projective if for a family of uzzy bitopological space $\{(X_i, s_i, t_i): i \in \Lambda\}$, the product fuzzy bitopological space $(\Pi X_i, \Pi s_i, \Pi t_i)$ has property P implies that each coordinate space has property P.

Definition A.12: [21] Let (X, T) be an ordinary topological space. The set of all lower semi continuous functions from (*X, T*) into the closed unit interval *I* equipped with the usual topology constitutive a fuzzy topology associated with (X, T) and is denoted as (X, $\omega(T)$).

Theorem A.1: [12] A bijective mapping from an fts (*X, t*) to an fts (*Y, s*) preserves the value of a fuzzy singleton (fuzzy point).