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Morse-Novikov Cohomology Computations on Low Dimensional Manifolds

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ABSTRACT

In literature, many researchers have exploited the concept of Lichnerowicz cohomology rings to study the topological invariants and properties of a manifold with differentiable structure. Lichnerowicz cohomology is also commonly known as Morse-Novikov cohomology which is defined as the ring of closed forms that are not exact by twicking the usual differential operator d by a closed $1-$form τ as $d + \tau\wedge$. This article is on the relationship between the computation of Morse-Novikov cohomology compared with de Rham cohomology on some low dimensional manifolds along with homotopy axiom, and Mayer-Vietoris sequence.

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1 Introduction

Let M be a manifold with differentiable structure of dimension k; denote by $\Lambda^p(M)$ the set of all degree p differential forms on M and the de Rham cohomology ring is denoted by $H^p(M)$. Let τ be a closed 1-form not necessarily exact forming the twisted operator $d_\tau = d + \tau\wedge : \Lambda^p(M) \to \Lambda^{p+1}(M)$. It can easily be varified that $d_\tau \circ d_\tau = 0$. The cochain complex $(\Lambda^*(M), d_\tau)$ of the manifold M is known as the Morse-Novikov complex. The Morse-Novikov or Lichnerowicz cohomology rings of M are the cohomology rings $H^k_\tau(M)$ of this cochain complex. Here, we summarize some results from the literature that shows the usefulness of Morse-Novikov cohomology. To study poisson geometry, Lichnerowicz in [1] studied the morse novikov cohomology at first. The zeros of the form τ has a combinatorial relation with ranks of these cohomologies which has been used to give a generalization of the Morse inequalities in [2] and [3]. An analytic proof of the real part of the Novikov's inequalities has been studied by Pazhintov [4]. Witten exploited exactness of τ to his famous invention *Witten deformation* in [5]. Shubin and Novikov applied the *Witten deformation* to a rigorous anlysis of limits of eigenvalues of Witten Laplacians for vector field and some more generalized 1-form in [6] and [7]. For 1-forms with non isolated zeros and vector fields Braverman and Farber [8] generalized them. See [9] for more on this topics. Otiman in [10] studied Lichnerowicz cohomology for special classes of closed 1-forms. An important

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result in this connection due to Chen in [11], proved that a Riemannian manifold M with almost non-negative sectional curvature and nontrivial first de Rham cohomology ring has trivial Morse-Novikov cohomology ring independent of the closed non-exact 1-form τ. In [12], Meng proved the Leray-Hirsch theorem for Morse-Novikov cohomology and for Dolbeault-Morse-Novikov cohomology on complex manifolds, a blowup formula. Locally conformal symplectic manifolds has also been studied using Morse-Novikov cohomology theory in [13], [14], and [15]. The sole purpose of this article is to compute Morse-Novikov cohomology of some low dimensional manifolds which is not found in literature usually. Using $d + \tau\wedge$ as the differential for a closed 1-form τ, we compute the Morse-Novikov cohomology rings compared to de Rham cohomology rings of some low dimensional standard differentiable manifolds whose isomorphism classes turn out to be homotopy invariants. This manuscript is composed from my doctoral thesis [16].

2 Morse-Novikov Cohomology

Let M be a manifold with differentiable structure of dimension k; denote by $\Lambda^p(M)$ the set of all degree p differential forms on M and the de Rham cohomology ring is denoted by $H^p(M)$. Let τ be a closed 1-form not necessarily exact forming the twisted operator $d_\tau = d + \tau\wedge : \Lambda^p(M) \to \Lambda^{p+1}(M)$. where d is the usual exterior derivative. Since $d \circ d = d^2 = 0$, $\tau \wedge \tau = 0$,and $d(\tau \wedge \nu) = d\tau \wedge \nu - \tau \wedge d\nu$ for any p-form ν, it can easily be varified that $d_\tau \circ \tau = 0$. The cochain complex $(\Lambda^*(M), d_\tau)$ of the manifold M is known as the Morse-Novikov complex. The Morse-Novikov or Lichnerowicz cohomology rings of M are the cohomology rings $H^k_\tau(M)$ of this cochain complex. Let d^τ_p be the restriction of d_τ to $\Lambda^p(M)$. The cohomology ring is defined as

$$H^p_\tau(M) = \frac{ker(d^\tau_p)}{im(d^\tau_{p-1})}.$$

This cohomology ring is also known as Lichnerowicz cohomology ring [1].

Example 1. *Let S^1 be the circle with its standard differentiable structure, consider $d_\tau = d + \tau\wedge$, where $[\tau] = d\theta \in \Lambda^1(S^1)$, and ϕ be a d_τ closed zero form (Zeroth forms are all differentiable functions on a manifold). Since ϕ is d_τ closed, $d_\tau\phi = 0$. Then*

$$\begin{aligned} d\phi + \phi\theta &= 0 \\ \Rightarrow \phi &= \mu e^{-\theta}, \end{aligned}$$

where μ is an integrating constant. Since all functions of the circle is periodic, $f = \mu e^{-\theta}$ does not define a function unless $\mu = 0$. Therefore $\ker(d_\tau) \subset \Lambda^0(S^1)$ is empty. Hence $H^0_\tau(S^1)$ is trivial.
Since $\Lambda^2(S^0)$ is empty, any $1-$form on S^1 is d_τ-closed. Let $\phi(\theta)\, d\theta \in \tau^1(S^1)$ be exact. Then there exists zero form ψ such that $d_\tau\psi = \phi(\theta)d\theta$. To solve this equation we may consider the fourier series expansions $\psi(\theta) = \sum \alpha_m e^{im\theta}$ and $\phi(\theta) = \sum \beta_n e^{in\theta}$. It turns out that $\psi(\theta) = \sum \frac{\beta_n}{in+1} e^{in\theta}$, and $\phi(\theta)d(\theta)$ is d_τ exact. Notice If ϕ has rapidly decreasing coefficients in its Fourier expansions, so has ψ. Hence $H^1_\tau(S^1)$ is also a trivial ring. Since $\Lambda^p(S^1)$ is empty for all $p \geq 2$,then $H^p_\tau(S^1)$ is trivial for all $p \geq 2$.

Remark 1. *It is known that S^1 has almost non-negative sectional curvature, from the main theorem of [12], we may conclude that all Morse-Novikov cohomology of circle vanish.*

Proposition 1. *If τ and $\mu = \tau + d\phi$ are cohomologous in $H^1(M)$, For each p, the map $[\mu] \mapsto [e^{-\phi}\mu]$ is an isomorphism of Morse-Novikov cohomology rings $H^p_\tau(M)$ and $H^p_\theta(M)$*

$$H^p_\tau(M) \cong H^p_\mu(M).$$

Remark 2. *If the manifold is equipped with a Riemannian structure with a particular metric on the tangent bundle, it is sufficient to choose the harmonic representative τ in a cohomology $[\alpha]$ class to compute all possible $H^k_\alpha(M)$, for every closed form is cohomologous to a harmonic form, one could always pick the harmonic representative [17].*

Proof. Then there exists $\psi \in H^0(M)$, such that $\mu - \tau = d\psi$. The map $\phi : H^p_\tau(M) \to H^p_\mu(M)$ defined as $\phi([\alpha]) = [e^{-\psi}\alpha]$ is well-defined ring homomorphism. since

$$(d + \mu\wedge)\left(e^{-\psi}\alpha\right) = (d + \tau \wedge + d\psi\wedge)\left(e^{-\psi}\alpha\right)$$

$$= \quad e^{-\psi}\left(d+\tau\wedge\right)(\alpha),$$

for all $\alpha \in H^p_\tau(M)$.

Suppose α,$\beta\in H^p_\tau(M)$ are cohomologous. Then there exists $\nu \in H^{p-1}_\tau(M)$ such that $\alpha-\beta=(d+\tau\wedge)\nu$. We have

$$\begin{aligned} \phi([\alpha-\beta]) &= [e^{-\psi}(d+\tau\wedge)\nu] \\ \Rightarrow \left[(d+\mu\wedge)\left(e^{-\psi}\nu\right)\right] &= [0]. \end{aligned}$$

Similarly if $\phi([\alpha])=0$, Then $[\alpha]=0\in H^k_\tau(M)$, and ϕ is one to one.
If $[\alpha]\in H^k_\mu(M)$ then we find similarly that $\left[e^{\psi}\alpha\right]\in H^p_\tau(M)$, so that ϕ onto. □

Corollary 1. *Let τ be an exact 1-form, the Morse-Novikov cohomology ring $H^p_\tau(M)$ and the de Rham cohomology ring $H^k(M)$ are isomorphic. i.e,*

$$H^k_\tau(M)\cong H^k(M),$$

for each p.

Corollary 2. *If all closed one forms on a manifold is exact, i.e, the first de Rham cohomology group $H^1(M)=0$, then for each p the Morse-Novikov cohomology rings satisfy $H^p_\tau(M)=H^p(M)$.*

Example 2. *Since the de Rham cohomology ring $H^1(S^n)$ vanishes, For $n\geq 2$, the Morse-Novikov cohomology $H^p_\tau(S^n)=H^k(S^n)$ for each p. The same result $H^p_\tau(\mathbb{R}^n)=H^p(\mathbb{R}^n)$ for $p>1$ holds for $\mathbb{R}^n$.*

Corollary 3. *The abelianization of first fundamental group of a manifold $\pi_1(M)$ is isomorphic to the de Rham cohomology group of the manifold. If it is finite, Morse-Novikov cohomology rings are isomorphic to the de Rham cohomology rings.*

Proof. The abelianization of $\pi_1(M)$ is isomorphic to the homology group $H_1(M,\mathbb{Z})$. The universal coefficient theorem for singular cohomology with integer coefficients states that the singular cohomology $H^1(M,\mathbb{R})$ of the manifold M is isomorphic to $\mathrm{Hom}\left(H_1\left(M,\mathbb{Z}\right),\mathbb{R}\right)$. A homomorphism into the set of integers from a finite group is the zero homomorphism, and refuging to the de Rham theorem $H^1(M,\mathbb{R})=H^1(M)\cong 0$, we have $H^1(M,\mathbb{R})=0$. □

Example 3. *Since the first homotopy group of real projective space is finite*

$$\pi_1(\mathbb{R}P^n)=\mathbb{Z}_2$$

, then $H^k_\tau(\mathbb{R}P^n)\cong H^k(\mathbb{R}P^n)$ for all k and any closed 1-form τ.

Lemma 1. *Let M be a smooth, τ is not exact if and only if the Morse-Novikov cohomology $H^0_\tau(M)=\{0\}$.*

Proof. Assume $H^0_\tau(M)\neq\{0\}$ for a closed 1-form τ. There exists a nonzero function $\phi\in C^\infty(M)$ for which we have

$$\begin{aligned} d_\tau\phi &= 0 \\ \Rightarrow d\phi+\phi\tau &= 0 \\ \Rightarrow d\left(\log(\frac{1}{\phi})\right) &= \tau. \end{aligned}$$

Therefore τ is exact. On the other hand, assume that τ is exact. There is a function $\psi\in C^\infty(M)$ such that $d\psi=\tau$. Then we have

$$d_\tau\left(e^{-\psi}\right) \quad = \quad -e^{-\psi}dg+e^{-\psi}d\psi=0,$$

hence we conclude $H^0_\tau(M)\neq\{0\}$. □

Example 4. *Computation of the Morse-Novikov cohomology rings of the torus $\mathbb{T}^2=\{(x,y)\in\mathbb{R}^2\}/2\pi\mathbb{Z}^2$ using the idea of periodic functions and their fourier series expansion. Suppose an 1-form τ that is closed but not exact. We know the de Rham cohomology ring of torus is spanned by the differential forms $\{1,dx,dy,dx\wedge dy\}$, where x and y are coordinates on the torus, with coefficients in $\mathbb{R}$, and $H^1(\mathbb{T}^2)=\{[\alpha dx+\beta dy]\,|\alpha,\beta\in\mathbb{R}\}$.*

We present a detail computation of $H_\tau^k(\mathbb{T}^2)$ for $\tau = dx$. For an arbitrary $\tau = \alpha dx + \beta dy$, the computation is analogous but requires huge manipulations.
If $\phi \in H_\tau^0(\mathbb{T}^2)$ is closed, i.e, $d_\tau\phi = 0$, then

$$d\phi + \phi dx = 0 \Rightarrow \phi = \mu e^{-x}$$

Notice $\phi(x, y) = ce^{-x}$ is a periodic function only when $c = 0$. Therefore, we have no d_τ closed function on $\mathbb{T}^2$. whence $H_\tau^0(\mathbb{T}^2)$ is trivial.
For the computation of $H_\tau^0(\mathbb{T}^2)$ consider the following functions defined in fourier series expantions

$$f(x, y) = \sum_{m,n\in\mathbb{Z}} f_{rsn} e^{i(rx+sy)},$$

$$\alpha(x, y) = \sum_{r,s\in\mathbb{Z}} \alpha_{rs} e^{i(rx+sy)},$$

and

$$\beta(x, y) = \sum_{r,s\in\mathbb{Z}} \beta_{rs} e^{i(rx+sy)}$$

in $H_\tau^0\left(\mathbb{T}^2\right)$. If $\chi = \alpha dx + \beta dy \in H_\tau^1(T^2)$ and $d_\tau\chi = 0$ then

$$\begin{aligned} -\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x} + \beta &= 0 \\ -in\alpha_{rs} + ir\beta_{rs} + \beta_{rs} &= 0 \\ \beta_{rs} &= \frac{is\alpha_{rs}}{ir + 1}. \end{aligned}$$

Particularly $\beta_{r0} = 0$ for each r.
Notice The differential equation $d_w\phi = \chi = \alpha dx + \beta dy$ has a solution whenever

$$\begin{aligned} \sum_{r,s\in\mathbb{Z}} (irf_{rs} + f_{rs} - \alpha_{rs}) e^{i(rx+sy)} dx + \sum_{r,s\in\mathbb{Z}} (isf_{rs} - \beta_{rs}) e^{i(rx+sy)} dy &= 0 \\ \Rightarrow irf_{rs} + f_{rs} - \alpha_{rs} &= 0, \end{aligned}$$

and $isf_{rs} - \beta_{rs} = 0$ for each pair of integers $r, s \in \mathbb{Z}$. It follows that $\phi_{rs} = \frac{\alpha_{rs}}{ir+1} = \frac{\beta_{rs}}{is}$, for $n \neq 0$. Therefore the one form $\chi = \alpha dx + \beta dy$ is d_τ exact. Hence $H_\tau^1(\mathbb{T}^1) = 0$.

For the computation of $H_\tau^2(\mathbb{T}^2)$, assume $\omega = \eta dx \wedge dy \in H_\tau^2(\mathbb{T}^2)$ is any 2-form for some smooth function η. Notice $d_\tau\omega = 0$. If ω is $d_\tau-$exact, then there is an $1-$form $\chi = \alpha dx + \beta dy$ such that $d_\tau\chi = \omega$, for some smooth functions α and β. It follows that $d_\tau\chi = \omega$ if and only if

$$\begin{aligned} \left(-\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x} + \beta - \nu\right) dx \wedge dy &= 0 \\ \Rightarrow -\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x} &= \eta - \beta \\ \Rightarrow d\chi &= \gamma dx \wedge dy, \end{aligned}$$

where $\gamma = \eta - g$. Then Stokes theorem states that $d\tau = \gamma dx \wedge dy$ have global solution on $\mathbb{T}^2$ if the integral of $\eta - g$ vanishes, which is obtained by choosing the function $\beta = \dfrac{\int_{\mathbb{T}^2} \eta}{\mathrm{vol}(\mathbb{T}^2)}$. Hence $H_\tau^2(\mathbb{T}^2) = 0$. Since $\mathbb{T}^2$ is two dimensional $H_\tau^1(\mathbb{T}^p) = 0$ for all $p \geq 3$.

Remark 3. *It is also known that $\mathbb{T}^2$ has almost non-negative sectional curvature, from the main theorem of [12], we may conclude that all Morse-Novikov cohomology rings of $\mathbb{T}^2$ are trivial.*

Definition 1. *Assume N and M are smooth manifolds and $I = [0, 1]$ is the unit interval. The smooth maps $\phi, \psi : N \to M$ are smooth homotopic, if there exists a smooth map $G : N \times I \to M$ so that $G(y, 0) = (y)$ and $G(y, 1) = \psi(y)$. In other words one map smoothly deforms in to the other map.*

Example 5. *Suppose* $,\psi : N \to \mathbb{R}^n$ *are smooth maps, and define* $G(y,\tau) : M \times \mathbb{R} \to \mathbb{R}^n$ *as* $G(y,\tau) = (1-\tau)\phi(y) + \tau\psi(y)$. *It is easy to notice that the map* G *defines a homotopy between the maps* ϕ *and* ψ. *Therefore they are smooth homotopic. This homotopy is usually known as straight line homotopy in Euclidian spaces.*

Definition 2. *A smooth homotopy equivalence between two manifolds* N, M *is a pair smooth maps* $\phi : N \to M$ *and* $\psi : N \to M$ *such that* $\phi \circ \psi$ *is homotopic to identity map of* M *and* $\psi \circ \phi$ *is homotopic to the identity map of* M. *In this case* M *and* N *are called homotopy equivalent, or they are of the same homotopy type.*

Notice that the binary relation defined on the set of all smooth manifolds by the homotopy type is an equivalence relation. Diffeomorphic manifolds are in the same homotopy class and the homotopy is induced from the diffeomorphism.

Example 6. *The space* $\mathbb{R}^{n+1} - \{0\}$ *and the sphere* S^n *have the same homotopy class. Consider* $i : S^n \to \mathbb{R}^{n+1} - \{0\}$ *is the inclusion map and* $\theta : \mathbb{R}^{n+1} - \{0\} \to S^n$ *is given as* $\theta(y) = \frac{y}{\|y\|}$. *Then obviously* $\theta \circ i$ *is the identity map of* S^n *and straight line provides a homotopy between* $i \circ \theta$ *and the identity of* $\mathbb{R}^{n+1} - \{0\}$. *More over notice that* θ *and* i *are inverses of one to the other.*

Let $\theta : M \to N$ be a map., and $f : N \to \mathbb{R}$ be a function. The pullback $f^* : M \to \mathbb{R}$ of the map f is defined by $f^*(x) = f(\theta(x))$. It can be shown that the standard differential operator pullback commutes with d. In other words $df^* = f^*d$. In a similar way the pull back of differential form is defined.

Proposition 2. *If* $G : M \to N$ *is smooth,* $\tau \in H^1(N)$, *and* $\tau \in H^k_\tau(N)$, *then* $G^* d_\tau \tau = d_{F^*\tau} F^* \tau$.

Proof. Let τ be a smooth k-form on N. Then

$$\begin{aligned} d_{G^*\tau} G^*\tau &= d\left(G^*\tau\right) + G^*\tau \wedge G^*\tau \\ &= G^*\left(d\tau\right) + F^*\left(\tau \wedge \tau\right) \\ &= G^*\left(d\tau + \tau \wedge \tau\right) = G^* d_\tau \tau. \end{aligned}$$

□

Since d commutes with pull back, it can be shown that a smooth map $\phi : N \to M$ pulls back a closed form into a closed form, and an exact form into an exact form.

Proposition 3. *Let* $\phi : N \Rightarrow M$ *be smooth, the pullback* $\phi^* : H^p(M) \to H^p(M)$ *provides a linear map* $\phi^* : H^k_\tau(M) \to H^k_{\phi^*\tau}(M)$ *in Morse-Novikov cohomology, defined by* $\phi^*([\tau]) = [\phi^*\tau]$.

Proof. By its definition ϕ^* is linear. We have to show that the map ϕ^* maps closed forms to closed forms and exact forms to exact forms. But we have

$$\begin{aligned} d_{\phi^*\tau}\left(\phi^*\tau\right) &= \phi^* d_\tau \tau \\ &= 0, \end{aligned}$$

for any $[\tau] \in H^k_\tau(N)$. The rest of the conclusion of the proposition follows from this. □

We are in a position to describe the smooth homotopy invariance of Morse-Novikov cohomology. The homotopy axiom for the Morse-Novikov cohomology.

Proposition 4. *Suppose* $\phi : N \to M$, *and* $\psi : N \to M$ *are homotopic, and* τ *be a closed* 1*-form on* M. *There exists a positive function* $\mu : N \to \mathbb{R}$ *such that*

$$\phi^* = \mu\psi^* : H^p_\tau(M) \to H^k_{\phi^*\tau}(M),$$

for all p.

Proof. Since there is a homotopy between the maps ϕ and ψ, homotopy axiom of de Rham cohomology provides that they induce the same map in de Rham cohomology. Hence the pull back $\phi^*\tau$,$\psi^*\tau \in H^1(N)$ of the closed 1-form $\tau \in H^1(N)$ are cohomologous. And there is a function $\nu : N \to \mathbb{R}$ such that $\psi^*\tau - \phi^*\tau = d\nu$. Define $\mu = e^\nu$. From the Proposition 1, it follows that $[\mu\psi^*\alpha] = [\phi^*\alpha] \in H^k_{\phi^*\tau}(N)$, for a d_τ closed form α on M. □

Corollary 4. *If $\phi : N \to M$ is a smooth homotopy and 1-form τ is closed , then the Morse-Novikov cohomology rings $H^*_\tau(N)$ and $H^*_{f^*\tau}(M)$ are isomorphic.*

$$H^p_\tau(N) \cong H^p_{f^*\tau}(M),$$

for all p.

Proof. We have the following linear maps

$$H^*_\tau(N) \xrightarrow{\phi^*} H^*_{\phi^*\tau}(M) \xrightarrow{\psi^*} H^*_{psi^*\phi^*\tau}(N).$$

By the homotopy axiom of Morse-Novikov cohomology, a positive function $\mu : N \to \mathbb{R}$ gives $\psi^*\phi^*\tau = \tau + d(\ln(\mu))$. It follows

$$\mathbb{I}_M = \mu\psi^*\phi^* = \mu\,(f \circ \psi)^* : H^*_\tau(N) \to H^*_\tau(N).$$

Similarly, a positive function $\bar{\mu} : M \to \mathbb{R}$ gives $\phi^*\psi^*\tau = \tau + d(\ln(\bar{\mu}))$. It follows

$$\mathbb{I}_N = \bar{\mu}\phi^*\psi^* = \mu\,(\psi \circ \phi)^* : H^*_\tau(M) \to H^*_\tau(M).$$

Hence ϕ^* and ψ^* are isomorphisms for multiplication by a positive function is an isomorphism of Morse-Novikov cohomology. □

Corollary 5. *Let G be a deformation retraction from M to S, and $\lambda : M \to S$ be the retraction $\lambda(x) = G(y, 1)$, where S be a submanifold of the manifold M. Then $\lambda^* : H^*_\tau(S) \to H^*_{\lambda^*\tau}(M)$ is an isomorphism in Morse-Novikov cohomology induced from λ.*

Proposition 5. *Mayer-Vietoris sequence of Morse-Novikov cohomology.*

Proof. Since the restriction map to open subsets of a manifold commutes with the differential d_τ, we can construct a Mayer-Vietoris long exact cohomology sequence for the Morse-Novikov cohomology in a similar fashion for de Rham cohomology which connects the cohomologies of the manifold with that of the open submanifolds. Suppose $M = P \cup Q$, where P, Q are open submanifolds of M. Then it can be shown that the following cochain complexes.

$$0 \to (\tau^*(M), d_\tau) \xrightarrow{\alpha} \left(\tau^*(P) \oplus \tau^*(Q),\, d_\tau|_P \oplus d_\tau|_Q\right) \xrightarrow{\beta} \left(\tau^*(P \cap Q),\, d_\tau|_{P\cap Q}\right) \to 0$$

are short exact sequence, where α is the restriction homomorphism induced from the inclusion $i : P \to M$ and β is the difference homomorphism induced from the inclusion map $j : P \cap Q \to M$,

$$\alpha(\tau) \mapsto \left(i^*_P\tau, i^*_Q\tau\right),$$

and

$$\beta\,(\tau, \eta) = j^*|_Q\,\eta - j^*|_P\,\tau.$$

[18] Homological algebra proves that a short exact sequence of cochain complexes provides a long exact sequence in cohomologies. Let ρ_P, ρ_Q is the partition of unity for the open cover U, V, the following long exact sequence

$$\cdots \to H^p_\tau(M) \xrightarrow{\alpha_*} H^p_{\tau|P}(P) \oplus H^p_{\tau|Q}(Q) \xrightarrow{\beta_*} H^*_{\tau|(P\cap Q)} \xrightarrow{\delta^*} H^{p+1}_\tau(M) \to \cdots$$

Mayer-Vietoris sequence of Morse-Novikov cohomology holds. The connecting homomorphism are defined by $\delta^*[\tau] = [-d_\tau(\rho_Q\tau)]$ on P and $\delta^*[\tau] = [-d_\tau(\rho_P\tau)]$ on Q, and

$$\begin{aligned} \alpha_*([\tau]) &= ([\tau|_P], [\tau|_Q]), \\ \beta_*([\theta], [\eta]) &= [\theta|_{P\cup Q}] - [\eta|_{P\cup Q}], \end{aligned}$$

□

Remark 4. *Proposition 5 is pretty handy to compute the Morse-Novikov cohomology of a manifold from the cohomology of its open submanifolds that are simpler manifolds in the sanse that we may compute their cohomologies much easily.*

3 Conclusions

The computational techniques from this article may be useful to find the Morse-Novikov cohomology of manifolds which are not considered here. Specially, Homotopy axiom and Mayer-Vietoris long exact sequence can be very handy to compute cohomology of manifolds from that of the open submanifolds of the manifolds along with short exact sequence.

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