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# Poincare Duality of Morse-Novikov Cohomology on a Riemannian Manifold

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**ABSTRACT**

Morse-Novikov or Lichnerowicz cohomology groups of a manifold has been studied by researchers to deduce properties and invariants of manifolds. Morse-Novikov cohomology is defined using the twisted differential $d_\omega = d + \omega\wedge$, where $d$ is the usual differential operator on forms, and $\omega$ is a non-exact closed 1-form on the manifold. On a Riemanian manifold each Morse-Novikov cohomolgy class has unique harmonic representative, and has Poincare duality isomorphism. This isomorhism have been proved in many elegant ways in literature. In this article we provide yet another proof using ellepticity of a differential complex, Green's operator, and Hodge star operator which may be useful in other computations related to Morse-Novikov cohomology.



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# 1 Introduction

Let $M$ be a manifold with differentiable structure of dimension $n$; denote by $\Omega^k(M)$ the set of all degree $k$ differential forms on $M$ and the de Rham cohomology ring is denoted by $H^k(M)$. Let $\omega$ be a closed $1-$form not necessarily exact forming the twisted operator $d_\omega = d + \omega\wedge : \Omega^k(M) \to \Omega^{k+1}(M)$. It can easily be verified that $d_\omega \circ d_\omega = 0$. The cochain complex $(\Omega^*(M), d_\omega)$ of the manifold $M$ is known as the Morse-Novikov complex. The Morse-Novikov or Lichnerowicz cohomology groups of $M$ are the cohomology groups $H^k_\tau(M)$ of this cochain complex. To study poisson geometry, A. Lichnerowicz in [1] studied, the Morse-Novikov cohomology first. The zeros of the form $\omega$ has a combinatorial relation with ranks of these cohomologies which has been used to give a generalization of the Morse inequalities in [2] and [3], S. P. Novikov while gave an analytic proof of the real part of the Novikov's inequalities has been studied by Pazhintov [4]. E. Witten exploited exactness of $\tau$ to his famous invention of the the Morse-Novikov cohomology for exact $\tau$ in his famous discovery *Witten deformation* in [5]. M. Shubin and S. P. Novikov applied the *Witten deformation* to

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an rigorous anlysis of limits of eigenvalues of Witten Laplacians for vector field and some more generalized 1-form in [6] and [7]. For 1-forms with non isolated zeros and vector fields, Braverman and Farber [8] generalized them. See [9] for more on this topics. Alexandra Otiman in [10] studied Lichnerowicz cohomology for special classes of closed 1-forms. An important result in this connection due to X. Chen showed in [11], proved that a Riemannian manifold $M$ with almost non-negative sectional curvature and nontrivial first de Rham cohomology ring has trivial Morse-Novikov cohomology ring independent of the closed non-exact 1-form $\omega$. In [12], L. Meng proved the Leray-Hirsch theorem for Morse-Novikov cohomology and for Dolbeault-Morse-Novikov cohomology on complex manifolds, a blow up formula. Locally conformal symplectic manifolds has also been studied using Morse-Novikov cohomology theory (see [13], [14], and [15]). Morse-Novikov cohomology groups using $d+\omega\wedge$ as the differential for a closed 1-form $\omega$, on Riemannian manifold has nice properties like each cohomology class has unique harmonic representative and finite dimensional, and has Poincare duality isomorphism. This isomorhism have been proved in many elegant ways in literature. In this article we provide yet another proof using ellepticity of a differential complex, Green's operator, and Hodge star operator which may be useful in other computations related to Morse-Novikov cohomology. This manuscript is composed from a section of my doctoral thesis [16].

## 2 Review of known results

For a introduction to Morse-novikov cohomology see [16] [17]. Here we define it with few examples. Let $M$ be a manifold with differentiable structure of dimension $n$; denote by $\Omega^k(M)$ the set of all degree $k$ differential forms on $M$ and the de Rham cohomology ring is denoted by $H^p(M)$. Let $\omega$ be a closed 1-form not necessarily exact forming the twisted operator $d_\omega = d+\omega\wedge : \Omega^k(M) \to \Omega^{k+1}(M)$, where $d$ is the usual exterior derivative. Since $d\circ d = d^2 = 0$, $\omega\wedge\omega = 0$,and $d(\omega\wedge\nu) = d\omega\wedge\nu - \omega\wedge d\nu$ for any $k$-form $\nu$, it can easily be varified that $d_\omega \circ d_\omega = 0$. The cochain complex $(\Omega^*(M), d_\omega)$ of the manifold $M$ is known as the Morse-Novikov complex. The Morse-Novikov or Lichnerowicz cohomology rings of $M$ are the cohomology rings $H^k_\omega(M)$ of this cochain complex. Let $d^\omega_k$ be the restriction of $d_\omega$ to $\Omega^k(M)$. The cohomology group is defined as

$$H^k_\omega(M) = \frac{ker(d^\omega_k)}{Im(d^\omega_{k-1})}.$$

This cohomology group is also known as Lichnerowicz cohomology group [1].

**Example 1.** *[16][17] Morse-Novikov cohomology groups of $S^1$ are trivial.*

**Example 2.** *[16][17] Morse-novikov cohomology group of real projective space $H^k_\omega(\mathbb{R}P^n) \cong H^k(\mathbb{R}P^n)$ for all $k$ and any closed 1-form $\omega$. Where $H^k(\mathbb{R}P^n)$ is the de Rham Cohomology group.*

**Example 3.** *[16][17] Morse-Novikov cohomology groups of $\mathbb{T}^2 = \{(x,y)\in\mathbb{R}^2\}/2\pi\mathbb{Z}^2$ are trivial.*

We now review some well-known facts (see, e.g [18]). Let $(M,g)$ be a closed compact oriented Riemannian manifold of dimension $n$. At every point $p\in M$, we have an inner product $g_p$ on the tangent space $T_pM$, and therefore also an inner product on the cotangent space $T^*_pM$ determined by the inverse matrix of the matrix of $g_p$. This inner product is extended in a natural way to differential forms. So each vector bundle $\Lambda^k T^*M$ carries a metric that allows us to define an inner product on the space of smooth $k$-forms on $M$ by the following formula

$$\langle\alpha,\beta\rangle = \int_M g(\alpha,\beta)\mathrm{vol}.$$

Let $\alpha\in\Omega^k(M)$ be a $k$-form. Define the linear Hodge star operator $* : \Omega^k(M)\to\Omega^{n-k}(M)$ such that for all $\beta\in\Omega^k(M)$

$$\alpha\wedge *\beta = g(\alpha,\beta)\mathrm{vol}.$$

So the inner product defined above can be expressed by the even simpler formula

$$\langle\alpha,\beta\rangle = \int_M \alpha\wedge *\beta.$$

It turns out that $**\alpha = (-1)^{k(n+k)}\alpha$ for $\alpha\in\Omega^k(M)$ and that $\beta\wedge *\alpha = \alpha\wedge *\beta$ for all $\alpha,\beta\in\Omega^k(M)$.

The codifferential $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ in the exterior algebra may be expressed in terms of the Hodge $*$ operator; for $\beta \in \Omega^k(M)$,

$$d^*\beta = (-1)^{nk+n+1} * (d * \beta).$$

**Lemma 1.** *(See, for example, [18]) On a closed compact Riemannian manifold, $d^*$ is the formal adjoint of $d$ with respect to the global inner product defined above.*

It follows that $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is an isomorphism. Since $*$ commutes with $\Delta = d^*d + dd^*$, $*$ is the Poincaré duality isomorphism of de Rham cohomology of a compact oriented manifold,

$$H^k(M) \cong H^{n-k}(M) \text{ for every } 0 \leq k \leq n.$$

The interior product in the exterior algebra is defined in terms of the Hodge $*$ operator; for $\beta \in \Omega^k(M)$ and $\omega \in \Omega^1(M)$ is a covector, the interior product $\omega \lrcorner : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined as

$$\omega \lrcorner \beta = (-1)^{nk+n} * (\omega \wedge *\beta).$$

**Lemma 2.** *The adjoint of $\omega\wedge$ with respect to the inner product defined above is $\omega\lrcorner$.*

*Proof.* Let $\beta \in \Omega^k(M)$ and $\gamma \in \Omega^{k-1}(M)$, then

$$\begin{aligned}(\gamma, \omega\lrcorner\beta)\,\mathrm{vol} &= (-1)^{nk+n}\,\gamma \wedge **(\omega \wedge *\beta)\\ &= (-1)^{nk+n+(n-k+1)(-k+1)}\,\gamma \wedge \omega \wedge *\beta\\ &= (-1)^{k+1}(-1)^{k-1}\,\omega \wedge \gamma \wedge *\beta\end{aligned}$$

so that $(\gamma, \omega\lrcorner\beta)\,\mathrm{vol} = (\omega \wedge \gamma, \beta)\,\mathrm{vol}$. □

Laplace and Dirac type operators [18], [19] are examples of elliptic operators. We first define the *principal symbol* of a differential or pseudodifferential operator. If $\pi : E \to M$ and $\pi' : F \to M$ are two vector bundles and $P : \Gamma(E) \to \Gamma(F)$ is a differential operator of order $k$ acting on sections, then in local coordinates of a local trivialization of the vector bundles $P$ can be written as

$$P = \sum_{|\alpha|=k} s_\alpha(x) \frac{\partial^k}{\partial x^\alpha} + \text{lower order terms},$$

where the summation is over all possible multi-indices $\alpha = (\alpha_1, \cdots, \alpha_k)$ of length $|\alpha| = k$ and each $s_\alpha(x) \in \mathrm{Hom}(E_x, F_x)$ is a linear transformation. If $\xi = \sum \xi_j dx^j \in T_x^*(M)$ is a non-zero covector at $x$, we define the *principal symbol* of $P$ to be

$$\sigma(P)(\xi) = i^k \sum_{|\alpha|=k} s_\alpha(x)\xi^\alpha \in Hom(E_x, F_x),$$

where $\xi^\alpha = \xi_{\alpha_1} \cdot \cdots \cdot \xi_{\alpha_n}$. It turns out that the principal symbol is invariant under coordinate transformations. One coordinate-free definition of $\sigma(P)_x : T_x^*M \to \mathrm{Hom}(E_x, F_x)$ can be given as follows. For any $\xi \in T_x^*M$ choose a locally defined function $f$ such that $df_x = \xi$. Then we define the operator

$$\sigma_m(P)(\xi) = \lim_{t\to\infty} \frac{1}{t^m}(e^{-itf} P e^{itf}),$$

where $(e^{-itf} P e^{itf})(u) = e^{-itf}(P(e^{itf}u))$. Then the order $k$ of the operator and symbol are defined to be $k = \sup\{m : \sigma_m(P)(\xi)\} < \infty$ and $\sigma(P)(\xi) = \sigma_k(P)(\xi)$. It follows that if $P$ and $Q$ are two differential operators such that the composition $PQ$ is defined, then

$$\sigma(PQ)(\xi) = \sigma(P)(\xi)\sigma(Q)(\xi).$$

**Definition 1.** *An elliptic differential operator $P$ on $M$ is defined to be an operator such that its principal symbol $\sigma(P)(\xi)$ is invertible for all nonzero covectors $\xi \in T^*M$.*

**Example 4.** *The symbol of the Dirac operator* $D = \sum c(e_j)\nabla_{e_j}$ *is*

$$\sigma(D)(\xi) = i\sum c(e_j)\xi_j = i\sum c(\xi^j e_j) = ic(\xi^\sharp).$$

*The symbol of the Dirac Laplacian* $D^2$ *is*

$$\sigma(D^2)(\xi) = \sigma(D)(\xi)\sigma(D)(\xi) = (ic(\xi^\sharp))^2 = \|\xi^\sharp\|^2,$$

*where* $\xi^\sharp$ *is the corresponding vector of the covector* $\xi$ *induced by the metric on* $M$. *The last equality is a consequence of the definition of Clifford multiplication; see [20]. Therefore for non-zero* $\xi$, *both these symbols are invertible, and hence* $D$ *and* $D^2$ *are elliptic differential operators.*
*An operator* $P$ *is strongly elliptic if there exists* $c > 0$ *such that*

$$\sigma(P)(\xi) \geq c|\xi|^2$$

*for all non-zero* $\xi \in T^*M$. *The Laplacian* $\Delta$ *of* $\mathbb{R}^n$ *and* $D^2$ *on Clifford bundle are strongly elliptic. For more about elliptic differential operators on manifolds see [18], [19], [20].*

# 3 Main result

Let $M$ be a closed, compact, and oriented Riemannian manifold. We consider the de Rham operator for the differential

$$d_w : \Omega^{e/o}(M) \to \Omega^{o/e}(M),$$

where $\Omega^e(M)$ and $\Omega^o(M)$ denote the bundle of differential forms of even degree and odd degree respectively. We choose a Riemannian metric $g$ on $M$; this induces a volume form on $M$ and Hermitian inner products on all the spaces $\Omega^k(M)$. Since $d_\omega$ is a linear differential operator and the bundle in question carries a Hermitian metric induced from the Hermitian inner product, there exists an unique adjoint of $d_\omega$, denoted by $d_\omega^*$. Combining $d_\omega$ and $d_\omega^*$ we obtain a deformed differential operator

$$D_\omega = d_\omega + d_\omega^* : \Omega^{e/o}(M) \to \Omega^{o/e}(M).$$

For each $k$, we define the *Laplace operator* $\Delta_\omega : \Omega^k(M) \to \Omega^k(M)$ by the formula $\Delta_\omega = (d_\omega + d_\omega^*)^2 = d_\omega d_\omega^* + d_\omega^* d_\omega$. A form $\tau \in \Omega^k(M)$ is called $\omega$-harmonic if $\Delta_\omega \tau = 0$. We denote $\mathscr{H}_\omega^k(M) = \ker\Delta_\omega$, the space of all $\omega$-harmonic forms of degree $k$. Notice that $\Delta_\omega$ is a second order, formally self adjoint, linear differential operator on $\Omega^k(M)$. Because $d_\omega$ and $d_\omega^*$ square to zero,

$$(\Delta_\omega\alpha, \beta) = (d_\omega\alpha, d_\omega\beta) + (d_\omega^*\alpha, d_\omega^*\beta) = (\alpha, \Delta_\omega\beta).$$

**Theorem 1.** $\ker\Delta_\omega$ *is finite dimensional.*

*Proof.* Since the principal symbols of $d_\omega + d_\omega^*$, and $\Delta_\omega$ are the same as that of $d + d^*$ and $\Delta$, the operators$d_\omega + d_\omega^*$ and $\Delta_\omega$ are elliptic operators. The following sequence

$$\Gamma\left(M, \Lambda^0(M)\right) \overset{d_\omega}{\to} \Gamma\left(M, \Lambda^1(M)\right) \overset{d_\omega}{\to} \cdots \overset{d_\omega}{\to} \Gamma\left(M, \Lambda^n(M)\right)$$

is an *elliptic complex*, since the associated symbol sequence

$$0 \to \pi^*\Gamma\left(M, \Lambda^0(M)\right) \overset{\sigma(d_\omega)}{\to} \cdots \overset{\sigma(d_\omega)}{\to} \pi^*\Gamma\left(M, \Lambda^n(M)\right) \to 0$$

is exact, where $\Gamma\left(M, \Lambda^k(M)\right) = \Omega^k(M)$ is the set of smooth sections of the bundle $\pi : \Lambda^k(M) \to M$, and $\sigma(d_\omega)$ is the principal symbol of $d_\omega$. See Chapter IV, Example 2.5 of [19]. We may therefore apply the theorem concerning an elliptic differential complex of vector bundles (see Chapter IV, Theorem 5.2 of [19]) to conclude that $\mathscr{H}_\omega^k(M) = \ker\Delta_\omega$ is finite dimensional, and we have the following orthogonal decomposition of $\Omega^k(M)$:

$$\Omega^k(M) = \mathscr{H}_\omega^k \oplus \operatorname{im}(\Delta_\omega G),$$

where $G : \Omega^k(M) \to \Omega^k(M)$ is a Green's operator. □

Now we can state and prove the Hodge theorem for the Morse-Novikov cohomology.

**Theorem 2.** *Let $(M,g)$ be a closed compact and oriented Riemannian manifold. Then $\mathscr{H}_{\omega}^{k}(M) \cong H_{\omega}^{k}(M)$. In other words, every Morse-Novikov cohomology class has a unique $\omega$-harmonic representative.*

*Proof.* Let $\alpha \in \mathscr{H}_{\omega}^{k}(M)$, which is smooth by elliptic regularity. Then we have

$$\begin{aligned}(\Delta_{\omega}\alpha, \alpha) &= 0 \\ \Rightarrow (d_{\omega}\alpha, d_{\omega}\alpha) + (d_{\omega}^{*}\alpha, d_{\omega}^{*}\alpha) &= 0 \\ \Rightarrow \|d_{\omega}\alpha\|^{2} + \|d_{\omega}^{*}\alpha\|^{2} &= 0.\end{aligned}$$

This implies that $\alpha$ is $\omega$-harmonic if and only if $d_{\omega}\alpha = 0$ and $d_{\omega}^{*}\alpha = 0$. These $\omega$-harmonic forms are closed and therefore define classes in Morse-Novikov cohomology. We have a map $\mathscr{I} : \mathscr{H}_{\omega}^{k}(M) \to H_{\omega}^{k}(M)$ defined by $\mathscr{I}(\alpha) = [\alpha]$. We show that this map is a bijection.

Suppose $\alpha \in \mathscr{H}_{\omega}^{k}$ is $d_{\omega}$ exact, say $\alpha = d_{\omega}\tau$ for some $\tau \in \Omega^{k-1}(M)$. Then

$$\|\alpha\|^{2} = (\alpha, \alpha) = (\alpha, d_{\omega}\tau) = (d_{\omega}^{*}\alpha, \tau) = 0,$$

and therefore $\alpha = 0$. To prove the surjectivity, let $\alpha \in \Omega^{k}(M)$ such that $d_{\omega}\alpha = 0$. Then by the decomposition $\Omega^{k}(M) = \mathscr{H}_{\omega}^{k} \oplus \operatorname{im}(\Delta_{\omega}G)$, for some $\tau \in \mathscr{H}_{\omega}^{k}(M)$ and $\beta \in \Omega^{k}(M)$, we have

$$\alpha = \tau + \Delta_{\omega}G\beta = \tau + d_{\omega}d_{\omega}^{*}G\beta + d_{\omega}^{*}d_{\omega}G\beta.$$

Applying $d_{\omega}$ on both sides of this equation, it follows that $d_{\omega}d_{\omega}^{*}d_{\omega}G\beta = 0$, and therefore

$$\|d_{\omega}^{*}d_{\omega}G\beta\|^{2} = (d_{\omega}^{*}d_{\omega}G\beta, d_{\omega}^{*}d_{\omega}G\beta) = (d_{\omega}G\beta, d_{\omega}d_{\omega}^{*}d_{\omega}G\beta)$$

proving that $d_{\omega}^{*}d_{\omega}G\beta = 0$. Hence we have $\alpha = \tau + d_{\omega}d_{\omega}^{*}G\beta$; therefore $[\alpha] = [\tau]$. □

Now we give a proof of Poincaré duality for Morse-Novikov cohomology, see Proposition 3.5 [21], using the Hodge star operator and the Hodge theorem for Morse-Novikov cohomology.

**Theorem 3.** *If $M$ is a closed compact oriented manifold of dimension $n$ and $\omega$ is a closed 1-form, then the Hodge star operator $* : \Omega^{k}(M) \to \Omega^{n-k}(M)$ induces the isomorphism*

$$H_{\omega}^{k}(M) \cong H_{-\omega}^{n-k}(M).$$

*Proof.* From $(\omega\lrcorner) = (-1)^{nk+n} * (\omega\wedge)*$, $*^{2} = (-1)^{k(n-k)}$, and $d^{*} = (-1)^{n(k+1)+1} * d*$ on $\Omega^{k}(M)$, we have the following identities for operators acting on $\Omega^{k}(M)$. For any $\beta \in \Omega^{k}(M)$

$$\begin{aligned}(\omega\lrcorner) * \beta &= (-1)^{n(n-k)+n} * (\omega\wedge) *^{2} \beta \\ &= (-1)^{n^{2}+nk+n} (-1)^{k(n-k)} * (\omega\wedge) \beta,\end{aligned}$$

so that $(\omega\lrcorner)* = (-1)^{k} * (\omega\wedge)$ on $\Omega^{k}(M)$. Also,

$$\begin{aligned}*(\omega\lrcorner)\beta &= (-1)^{nk+n} *^{2} (\omega\wedge) * \beta \\ &= (-1)^{nk+n} (-1)^{(n-k+1)(n-(n-k+1))} (\omega\wedge) * \beta,\end{aligned}$$

so that $*(\omega\lrcorner) = (-1)^{k+1} (\omega\wedge)*$ on $\Omega^{k}(M)$. Next

$$\begin{aligned}d^{*} * \beta &= (-1)^{n(n-k+1)+1} * d *^{2} \beta \\ &= (-1)^{n(n-k+1)+1} (-1)^{k(n-k)} * d\beta,\end{aligned}$$

so that $d^{*}* = (-1)^{k+1} * d$ on $\Omega^{k}(M)$. Finally

$$*d^{*}\beta = (-1)^{n(k+1)+1} *^{2} d * \beta$$

$$= (-1)^{n(k+1)+1} (-1)^{(n-k+1)(n-(n-k+1))} d * \beta,$$

so that $*d^* = (-1)^k d*$ on $\Omega^k(M)$. From these equations we have

$$(d^* + \omega\lrcorner)* = (-1)^{k+1} * (d - \omega\wedge)$$
$$(d + \omega\wedge)* = (-1)^k * (d^* - \omega\lrcorner)$$

on $\Omega^k(M)$. As before $d^*$ is the $L^2$ adjoint of $d$, and $\lrcorner$ represents interior product. It turns out that the $L^2$ adjoint of $d_\omega = d + \omega\wedge$ is $d_\omega^* = d^* + \omega\lrcorner$ and the Laplacian is $\Delta_\omega = (d_\omega + d_\omega^*)^2 = d_\omega d_\omega^* + d_\omega^* d_\omega = (d + \omega\wedge)(d^* + \omega\lrcorner) + (d^* + \omega\lrcorner)(d + \omega\wedge)$. If $\beta \in \Omega^k(M)$, then by the formulas above we have for all $\beta \in \Omega^k(M)$,

$$\begin{aligned} *\Delta_\omega\beta &= *(d+\omega\wedge)(d^*+\omega\lrcorner)\beta + *(d^*+\omega\lrcorner)(d+\omega\wedge)\beta \\ &= (-1)^{k-1}(d^*-\omega\lrcorner)*(d^*+\omega\lrcorner)\beta + (-1)^k(d-\omega\wedge)*(d+\omega\wedge)\beta \\ &= (-1)^{k-1}(-1)^k(d^*-\omega\lrcorner)(d-\omega\wedge)*\beta + (-1)^k(-1)^{k+1}(d-\omega\wedge)(d^*-\omega\lrcorner)*\beta \\ &= -\left((d^*-\omega\lrcorner)(d-\omega\wedge) + (d-\omega\wedge)(d^*-\omega\lrcorner)\right)*\beta \\ &= -\Delta_{-\omega}*\beta. \end{aligned}$$

Thus the operator $*$ maps $\omega$-harmonic forms to $(-\omega)$-harmonic forms, so from the Hodge theorem for the Morse-Novikov cohomology $*$ induces the required isomorphism. □

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